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One-Dimensional Hard Rod Caricature of Hydrodynamics

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We give here a rigorous deduction of the "hydrodynamic" equation which holds in the hydrodynamic limit, for a model system of one-dimensional identical hard rods interacting through elastic collisions. The equation should be considered as the analog of the Euler equation of real hydrodynamics. Owing to the degeneracy of the model, it is written in terms of a function $g(q, v, t)$ expressing the density of particles with velocity v at the point q at time t. For this equation we prove an existence and uniqueness theorem in some natural class of functions. Our main result is the proof that if $\{P^\epsilon, \epsilon > 0\}$ is a class of initial states which are homogeneous on a scale much less than ϵ^{-1}, and if the corresponding particle densities tend, as $\epsilon \to 0$, in the proper scale, to the initial hydrodynamic density $g_0(q, v)$, then, under some general assumptions on the states P^ϵ and on g_0, the particle densities of the evolved states at time $\epsilon^{-1}t$, tend as $\epsilon \to 0$ to the unique solution of the hydrodynamic equation with initial condition g_0. The proof is completed by exhibiting a large class of initial families $\{P^\epsilon, \epsilon > 0\}$ which possess the required properties.

KEY WORDS: Nonequilibrium statistical mechanics; hydrodynamic limit; Euler equation; one-dimensional hard rods.

1. INTRODUCTION

The problem of deriving hydrodynamic equations from the equations of motion of a particle system is one of the central problems in statistical mechanics, as is testified by the huge physical literature devoted to the explanation of its various aspects. It is now possible, in view of the recent

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developments of rigorous statistical mechanics, to face the problem at a mathematical level for some simplified models of particle interaction.

A previous attempt of a rigorous deduction of hydrodynamics should be mentioned here, namely, the important paper of Morrey,[(1)] who was the first, to our knowledge, to introduce and discuss the notion of "hydrodynamic limit." We shall now briefly describe the ideas involved, having in mind mainly infinite particle systems, for which one should assume that the dynamics is well defined.

The specific character of the hydrodynamic situation is that the initial probability distribution (initial state) depends on a small parameter ϵ, which characterizes the ratio of the typical microscopic and macroscopic lengths. More precisely, the initial state should be homogeneous on a microscale (i.e., the distribution almost goes into itself for translations of order much less than ϵ^{-1}), at least for its "essential parameters," and it should be nonhomogeneous for translations of order ϵ^{-1}. In investigating the dynamics of such a state, since the fact that the initial distribution is nonhomogeneous appears at a given point only after a time of the order ϵ^{-1}, it is convenient to introduce the change of variables $t = \epsilon\tau$, where t and τ denote the macroscopic and the microscopic times, respectively. It is expected that when ϵ goes to zero (hydrodynamic limit) the change in macroscopic time of the essential parameters of the state will be described by hydrodynamic equations, more precisely by the Euler equations for a nonviscous fluid. It is assumed usually that the essential parameters are the particle density, the average particle momentum, and the temperature, or any other set of quantities connected with them by a one-to-one transformation. The assumption that such a behavior actually takes place can be considered as the foundation of hydrodynamics. Usually it is explained by saying that locally, in microvolumes of diameter of order much less than ϵ^{-1}, the probability distribution is "almost" a Gibbs equilibrium distribution, and, on a macroscopic scale, where a point corresponds to an infinitely large microscopic volume, only the change in space and time of the parameters of the Gibbs distribution (which are determined by density, average momentum, and temperature) has to be taken into account.

Morrey was not able to carry out his program, and in order to get the Euler equations he was compelled to introduce some assumptions on the evolution of a large-particle system which, up to now, have not been proved. In our opinion the difficulties faced by Morrey are of a fundamental character. Apparently, it is impossible to deduce hydrodynamics without developing technical tools which allow also to prove the local Gibbs character of the states. In the simple case in which the initial state is translation invariant this is equivalent to the foundation of the fundamental

Boltzmann–Gibbs postulate of statistical mechanics in what is perhaps its most natural formulation (see Ref. 2). It is well known that this problem is very difficult and its solution is a task for the future.

Since the situation is so poorly understood, it is useful to consider some simplified, degenerate, models, which, however, have the unique advantage that for them the hydrodynamic limit can be submitted to mathematical treatment. Because of the connection mentioned above between hydrodynamics and convergence to equilibrium, we can hope to succeed only in the cases in which we control convergence to equilibrium. This determines the choice of the model which we consider in this paper, namely, one-dimensional hard rods with elastic collisions, for which convergence to equilibrium has already been studied in Ref. 3. We remark that similar motivations led recently to the investigation of the hydrodynamic limit for some models (perhaps even more caricatural) of stochastic dynamics.[4–6]

In considering degenerate models we must give up from the beginning the hope of obtaining hydrodynamic equations of the type which is traditional in physics. In fact the very set of the parameters, the change of which is described by hydrodynamic equations, differs from the usual one. Thus in the model of one-dimensional hard rods the equilibrium state is determined by a function $g(v)$ such that $g(v)\,dv$ gives the average number of particles with velocities in the interval $(v, v + dv)$ (see Ref. 3), and therefore the hydrodynamics will be described by a function $g(q, v, t)$ giving the macroscopic mass distribution in space and velocity of the "hard rod fluid" at time t. So the picture which appears here is far from real hydrodynamics and can be considered only as a caricature. Nevertheless the authors hope that the study of such a degenerate situation may lead to a better understanding of the real hydrodynamics, in the same way as the study of deformities in living organisms is useful in understanding the physiology of normal organisms.

We now describe, omitting technical details, the basic results of this paper.

We consider the dynamics of an infinite system of identical hard rods of length $d > 0$ on the line $\mathbb{R}^1$, which move inertially between collisions and at collision exchange velocities (elastic collisions). The precise definition of the dynamics in the phase space of an infinite system requires some additional restrictions on the initial point, which are discussed in detail in Ref. 3. We consider a family $\{P^\epsilon, \epsilon > 0\}$ of initial states, depending on the parameter ϵ, i.e., a family of probability measures on the phase space of infinite hard rods on the line, which are concentrated on the subset of phase space for which the dynamics is well defined. Therefore the initial state P^ϵ induces in a natural way the evolved state P^ϵ_τ at the microtime τ. The additional conditions on the initial states $\{P^\epsilon, \epsilon > 0\}$ for the hydrodynamic limit are the following. Let $k^{(1)}_{P^\epsilon}(q,v)$ denote the first correlation function of the state P^ϵ (i.e., the density of particles with velocity v at the point q). The main assumption is the existence of the hydrodynamic initial condition, i.e., of a function $g_0(q,v)$, smooth in q, such that

$$\lim_{\epsilon\to 0} k^{(1)}_{P^\epsilon}(\epsilon^{-1}q, v) = g_0(q,v), \qquad q, v \in \mathbb{R}^1$$

g_0 has the meaning of a macroscopic mass distribution (in the rescaling process the particle mass can be assumed to be equal to ϵ). Moreover we assume that the states P^ϵ possess some property of decay of the correlations at large distances. One might formulate the hypothesis that without an assumption of this type it is in general impossible to obtain a universal hydrodynamic picture, independent of the choice of the initial states. The particular form which this assumption takes in our paper is connected with the technique of the proof, and is a law of large numbers for some nonlinear functional on the phase space, which should be satisfied uniformly in ϵ. To convince the reader that such a condition is not too restrictive, we prove in Section 4 that it is satisfied by a large class of Gibbs states.

Let $k^{(1)}_{P^\epsilon_{\epsilon^{-1}t}}$ denote the first correlation function of the evolved state $P^\epsilon_{\epsilon^{-1}t}$. The main result of our paper consists in proving that the following limit exists:

$$\lim_{\epsilon\to 0} k^{(1)}_{P^\epsilon_{\epsilon^{-1}t}}(\epsilon^{-1}q, v) = g(q,v,t), \qquad q, v, t \in \mathbb{R}^1$$

and that the limit function $g(q,v,t)$ is, under some general assumptions on the initial condition $g_0(q,v)$, the unique solution (in some class of functions) of the equation

$$\begin{aligned} &\frac{\partial}{\partial t} g(q,v,t) + v\frac{\partial}{\partial q} g(q,v,t) \\ &\quad + d\frac{\partial}{\partial q}\left\{ g(q,v,t)\left[1 - d\int_{\mathbb{R}^1} dv''\, g(q,v'',t)\right]^{-1} \right. \\ &\qquad \left. \times \int_{\mathbb{R}^1} dv'\,(v-v')g(q,v',t)\right\} = 0 \end{aligned} \tag{1.1}$$

with the initial condition $g(q,v,0) = g_0(q,v)$. This equation has to be considered as the analog of the usual Euler equation for our model.

Equation (1.1) is known to physicists, and in "physical terms" it is easily derived (see Ref. 7). In order to get an intuitive guideline we give here in some detail a heuristic derivation of it which clarifies its physical sense. It is convenient to pass to a modified picture of the motion, in which the colliding rods exchange their positions and keep their velocities, so that in the new picture each particle travels with constant velocity, but from time to time makes jumps of length d forward or backward.

Microscopically $g(q,v,t)\,dv$ is interpreted as the average density of the particles with velocity in the interval $(v, v+dv)$, at the macroscopic point q, and at the macroscopic time t. The corresponding total particle density is given by

$$\sigma_g(q,t) = \int_{\mathbb{R}^1} dv\; g(q,v,t)$$

and the average distance between particles "at the macropoint q" is given by $[1 - d\sigma_g(q,t)][\sigma_g(q,t)]^{-1}$. The probability that the first particle to collide with a particle of a given velocity v has a velocity in the interval $(v', v' + dv')$ will be $[g(q,v',t)/\sigma_g(q,t)]\,dv'$. Since the relative velocity is $v - v'$, the probability that such a collision takes place in the time interval $(t, t+dt)$ is

$$\begin{aligned}&\left[1 - d\sigma_g(q,t)\right]^{-1}\sigma_g(q,t)\left[g(q,v',t)/\sigma_g(q,t)\right]|v - v'|\,dv'\,dt\\&\quad= \left[1 - d\sigma_g(q,t)\right]^{-1} g(q,v',t)|v - v'|\,dv'\,dt\end{aligned}$$

Therefore the average velocity with which a particle with velocity v moves is

$$\hat{v}(q,v,t) = v + d\left[1 - d\sigma_g(q,t)\right]^{-1}\int_{\mathbb{R}^1} dv'\,(v - v')g(q,v',t)$$

Therefore the "volume element" of the "hard rod fluid" with velocity v which at time t is in the interval $(q, q+dq)$, will be at time $t+dt$ in the interval $[q + \hat{v}(q,v,t)\,dt,\, q + dq + \hat{v}(q+dq,v,t)\,dt]$. Taking into account the law of mass conservation we find the equality

$$\begin{aligned}g(q,v,t)\,dq &= g\left[q + \hat{v}(q,v,t)\,dt, v, t+dt\right]\\&\qquad\times\left[dq + \hat{v}(q+dq,v,t)\,dt - \hat{v}(q,v,t)\,dt\right]\end{aligned}$$

which, taking into account only the first-order terms, gives the equation

$$\frac{\partial}{\partial t}\, g(q,v,t) + g(q,v,t)\frac{\partial}{\partial q}\hat{v}(q,v,t) + \hat{v}(q,v,t)\frac{\partial}{\partial q}\, g(q,v,t) = 0$$

which is equivalent to Eq. (1.1).

The paper is organized as follows. In Section 2 we give the necessary preliminary facts. Section 3 is devoted to the study of the limit equation (1.1). We prove there an existence and uniqueness theorem. In Section 4 we prove a theorem of convergence to the solution of Eq. (1.1) under the hydrodynamic limit procedure. In Section 5 we give examples of families of initial states for which the conditions of the convergence theorem hold.

The results of this paper have been announced in short notes.[8,9] The proof of the results on the equation of higher order announced there will be published in another paper.

2. NOTATIONS AND PRELIMINARY FACTS

In this section we introduce the basic notations and facts. A more detailed account of the topics touched here can be found in Ref. 3.

By $\mathbb{Z}$ and $\mathbb{Z}_+$ we denote, respectively, the integers and the nonnegative integers. By $\mathbb{R}^1_+$ and $\mathbb{R}^1_-$ denote the nonnegative and negative real numbers. $\overline{\mathbb{R}}^1 = \mathbb{R}^1 \cup \{+\infty\}$ is the extended real line and $\overline{\mathbb{R}}^1_+ = \mathbb{R}^1_+ \cup \{+\infty\}$ is its positive part. If A is a finite or countable set we denote by $|A|$ its cardinality. If I is an interval, we denote by $|I|$ its length. Given a set $A \subset \mathbb{R}^\nu$, we denote by ∂A its boundary, and by A^c its complement $\mathbb{R}^\nu \backslash A$. Given two subsets $A, B \subset \mathbb{R}^\nu$ we denote their distance by $\operatorname{dist}(A, B) = \inf_{x \in A, y \in B} \|x - y\|$. For $A \subset \mathbb{R}^\nu$ and $x \in \mathbb{R}^\nu$ we set $x + A = \{y \in \mathbb{R}^\nu : y - x \in A\}$, and, for $A \subset \mathbb{R}^\nu$ and $t \in \mathbb{R}^1 \backslash \{0\}$, $tA = \{y \in \mathbb{R}^\nu : t^{-1}y \in A\}$. Throughout this paper we set $\exp(-\infty) = 0$, $-\log 0 = +\infty$, and $a + \infty = +\infty$ for any $a \in \mathbb{R}^1$.

The letters N and M denote the configuration space $\mathbb{R}^\nu$ and the phase space $\mathbb{R}^\nu \times \mathbb{R}^\nu$ of a ν-dimensional classical particle, $\nu = 1, 2, \ldots$. A point in N is denoted by q and in M by (q, v), q being the particle position and v the particle velocity, respectively. If $C \subset \mathbb{R}^\nu$ we denote by $M(C)$ the phase space $C \times \mathbb{R}^\nu$ of a particle in the "volume" C. Here and in the following all subsets of $\mathbb{R}^\nu$ and of $\mathbb{R}^\nu \times \mathbb{R}^\nu$ which we introduce are supposed to be measurable.

Let $\mathcal{N}$ (resp., $\mathcal{M}$) denote the collection of subsets $Y \subset N$ ($X \subset M$) such that for any bounded $C \subset \mathbb{R}^\nu$ the intersection $Y \cap C$ ($X \cap C \times \mathbb{R}^\nu$) is finite. $\mathcal{N}$ is considered as the configuration space and $\mathcal{M}$ as the phase space of a locally finite particle system in $\mathbb{R}^\nu$. A point $Y \in \mathcal{N}$ (resp., $X \in \mathcal{M}$) is called a particle configuration (realization) in $\mathbb{R}^\nu$. Given $Y \in \mathcal{N}$ (resp., $X \in \mathcal{M}$) we shall often call the points $q \in Y$ [$(q, v) \in X$] "particles" of Y (X). The empty set, regarded as a point of $\mathcal{N}$ ($\mathcal{M}$), is denoted by Θ, and is called the vacuum configuration (realization).

For $x \in \mathbb{R}^\nu$ we define the space translation S_x on $\mathcal{N}$ (resp., $\mathcal{M}$) by

$$S_x Y = \{q \in N : q - x \in Y\} \qquad (S_x X = \{(q, v) \in M : (q - x, v) \in X\}) \tag{2.1}$$

Given $C \subset N$ (resp., $D \subset M$), we set $Y_C = Y \cap C$, $Y \in \mathcal{N}$ ($X_D = X \cap D$, $X \in \mathcal{M}$). By $\mathcal{N}(C)$ [resp. $\mathcal{M}(C)$], $C \subset \mathbb{R}^\nu$ we denote the configuration (phase) space of a particle system in the volume C:

$$\mathcal{N}(C) = \{Y \in \mathcal{N} : Y = Y_C\} \qquad [\mathcal{M}(C) = \{X \in \mathcal{M} : X = X_{C \times \mathbb{R}^\nu}\}]$$

Let $\mathcal{N}^0$ (resp. $\mathcal{M}^0$) denote the collection of the finite subsets of N (M). If $C \subset \mathbb{R}^\nu$ is bounded, then $\mathcal{N}(C) \subset \mathcal{N}^0$ [resp. $\mathcal{M}(C) \subset \mathcal{M}^0$]. $\mathcal{N}^0$ (resp. $\mathcal{M}^0$) can be written as a union

$$\mathcal{N}^0 = \bigcup_{n=0}^{\infty} N_n \qquad \left(\mathcal{M}^0 = \bigcup_{n=0}^{\infty} M_n\right) \tag{2.2}$$

where $N_0(M_0) = \{\Theta\}$ and N_n (M_n), $n > 1$, is the collection of the n-point subsets of N (M). Clearly, N_1 (resp. M_1) can be identified with N (M), and N_n (M_n), $n > 2$, with the image of the set

$$N^n_{\neq} = \{(q_1, \ldots, q_n) \in N^n : q_i \neq q_j \text{ for } 1 < i < j < n\}$$

$$\big(M^n_{\neq} = \{[(q_1, v_1), \ldots, (q_n, v_n)] \in M^n :$$

$$(q_i, v_i) \neq (q_j, v_j) \text{ for } 1 < i < j < n\}\big)$$

under the symmetrization map Π_n:

$$\Pi_n(q_1, \ldots, q_n) = \bigcup_{i=1}^{n} \{q_i\} \qquad \left(\Pi_n[(q_1, v_1), \ldots, (q_n, v_n)] = \bigcup_{i=1}^{n} \{(q_i, v_i)\}\right)$$

The usual topology on N^n (resp. M^n), $n > 2$, induces, via the map Π_n, a topology on N_n (M_n). The Borel σ algebra of subsets of N_n (resp. M_n) is denoted by $\mathscr{C}_n$ $(\mathscr{D}_n)$. The Lebesgue measure m^n (resp. l^n) on N^n (M^n) induces, via Π_n, a measure on $(N_n, \mathscr{C}_n)$ $[(M_n, \mathscr{D}_n)]$, which we denote by m_n (l_n):

$$m_n(A) = \frac{1}{n!} m^n(\Pi_n^{-1} A), \qquad A \in \mathscr{C}_n$$

$$\left[l_n(A) = \frac{1}{n!} l^n(\Pi_n^{-1} A), \qquad A \in \mathscr{D}_n \right]$$

For $n = 1$ we have $\mathscr{C}_1 = \mathscr{C}$ (resp. $\mathscr{D}_1 = \mathscr{D}$) and $m_1 = m$ $(l_1 = l)$, where $\mathscr{C}$ and m ($\mathscr{D}$ and l) are the Borel σ algebra of subsets of N (M) and the Lebesgue measure on $(N, \mathscr{C})$ $[(M, \mathscr{D})]$, respectively. For $n = 0$ we set m_0 $(N_0) = 1$ $[l_0(M_0) = 1]$.

Let $\{G_\tau,\ \tau \in \mathbb{R}^1\}$ be a family of nonnegative measures on the space $(M, \mathscr{D})$ taking finite values on $M(C)$ for any bounded $C \subset \mathbb{R}^\nu$. We say that G_τ converges to a measure G in the vague topology as $\tau \to \tau_0$ (τ_0 can be $\pm\infty$) if for any bounded $C \subset \mathbb{R}^\nu$ and any bounded continuous function $f: M \to \mathbb{R}^1$ with support in $M(C)$

$$\lim_{\tau \to \tau_0} \int_M G_\tau(dq \times dv)\, f(q, v) = \int_M G(dq \times dv)\, f(q, v)$$

In case G is absolutely continuous with respect to l, this is equivalent to the fact that for any bounded parallelepipeds $I, J \subset \mathbb{R}^\nu$ with the edges parallel to the coordinate axes

$$\lim_{\tau \to \tau_0} G_\tau(I \times J) = G(I \times J)$$

The space $\mathcal{N}^0$ (resp. $\mathcal{M}^0$) is equipped with the topology of the topological sum with respect to the representation (2.2). The Borel σ algebra of subsets of $\mathcal{N}^0$ (resp. $\mathcal{M}^0$) is denoted by $\mathfrak{N}^0$ ($\mathfrak{M}^0$). The induced "Lebesgue" measure on $(\mathcal{N}^0, \mathfrak{N}^0)$ [resp. $(\mathcal{M}^0, \mathfrak{M}^0)$] is denoted by $\mu(\lambda)$:

$$\mu(A) = \sum_{n=0}^{\infty} m_n(A \cap N_n), \qquad A \in \mathfrak{N}^0$$
$$\left[\lambda(A) = \sum_{n=0}^{\infty} l_n(A \cap M_n), \qquad A \in \mathfrak{M}^0\right] \tag{2.3}$$

The measure element of μ (resp. λ) in integrals with respect to this measure will be simply denoted by dY (dX).

Given a bounded $C \subset \mathbb{R}^\nu$, consider the σ algebra $\mathfrak{N}(C) = \{A \subseteq \mathcal{N}(C) : A \in \mathfrak{N}^0\}$ [resp. $\mathfrak{M}(C) = \{A \subseteq \mathcal{M}(C) : A \in \mathfrak{M}^0\}$]. Denoting by π_C the projection map $Y \in \mathcal{N} \to Y_C$ (resp. $X \in \mathcal{M} \to X_{C \times \mathbb{R}^\nu}$) the σ algebra $\mathfrak{N}_C = \{\pi_C^{-1}A : A \in \mathfrak{N}(C)\}$ $[\mathfrak{M}_C = \{\pi_C^{-1}A : A \in \mathfrak{M}(C)\}]$ of subsets of $\mathcal{N}$ ($\mathcal{M}$) is isomorphic to $\mathfrak{N}(C)$ $[\mathfrak{M}(C)]$. We denote by $\mathfrak{N}$ ($\mathfrak{M}$) the smallest σ algebra of subsets of $\mathcal{N}$ ($\mathcal{M}$) containing $\mathfrak{N}_C$ ($\mathfrak{M}_C$) for any bounded $C \subset \mathbb{R}^\nu$. If $C \subset \mathbb{R}^\nu$ is unbounded we denote by $\mathfrak{N}_C$ ($\mathfrak{M}_C$) the smallest σ algebra containing $\mathfrak{N}_{C'}$ ($\mathfrak{M}_{C'}$) for any bounded $C' \subset C$.

For $E \subseteq N$ (resp. $E \subseteq M$) and $n \in \mathbb{Z}_+$ we set

$$A_{E,n} = \{Y \in \mathcal{N} : |Y \cap E| = n\} \qquad (A_{E,n} = \{X \in \mathcal{M} : |X \cap E| = n\})$$

Consider the topology of $\mathcal{N}$ (resp. $\mathcal{M}$) in which a fundamental system of neighborhoods of a given point Y (X) consists of the sets $A_{C,n}$ ($A_{C \times B,n}$) where $C \subset \mathbb{R}^\nu$ is an arbitrary bounded set (and $B \subseteq \mathbb{R}^\nu$ is an arbitrary set) such that $Y_{\partial C} = \Theta$ ($X_{\partial(C \times B)} = \Theta$) and where $n = |Y_C|$ ($n = |X_{C \times B}|$). It is well known (see, e.g., Ref. 10, Chap. 1.15) that $\mathcal{N}$ (resp. $\mathcal{M}$) is a polish space and $\mathfrak{N}$ ($\mathfrak{M}$) is the Borel σ algebra with respect to this topology.

An equivalent definition of the σ algebra $\mathfrak{N}$ (resp. $\mathfrak{M}$) is that $\mathfrak{N}$ ($\mathfrak{M}$) is generated by the family of random variables

$$\xi_C : Y \in \mathcal{N} \to |Y_C| \qquad (\xi_{C \times B} : X \in \mathcal{M} \to |X_{C \times B}|) \tag{2.4}$$

where $C \subset \mathbb{R}^\nu$ is an arbitrary bounded (and $B \subseteq \mathbb{R}^\nu$ is an arbitrary) set. Similarly, $\mathfrak{N}_C$ (resp., $\mathfrak{M}_C$) is generated by the family $\{\xi_{C'}\}$ ($\{\xi_{C' \times B'}\}$) where $C' \subset C$ is an arbitrary bounded (and $B' \subseteq \mathbb{R}^\nu$ is an arbitrary) set.

Definition 2.1. A probability measure P on $(\mathcal{N}, \mathfrak{N})$ [resp. $(\mathcal{M}, \mathfrak{M})$] is called a configuration (phase) state of a locally finite particle system in $\mathbb{R}^\nu$, or, shortly, a c state (state). The expectation value of a random variable f with respect to P is denoted by $\mathbb{E}_P f$, the variance by $\mathbb{D}_P f = \mathbb{E}_P(f - \mathbb{E}_P f)^2$. A c state (resp., state) P is called translation invariant if for any $A \in \mathfrak{N}$ ($A \in \mathfrak{M}$) and $x \in \mathbb{R}^\nu$

$$P(S_x A) = P(A) \tag{2.5}$$

Given a c state (resp., state) P, consider the measure K_P, on $(\mathcal{N}^0, \mathfrak{N}^0)$ $[(\mathcal{M}^0, \mathfrak{M}^0)]$ defined by

$$K_P(A) = \int_{\mathcal{N}} P(dY) \sum_{Y' \subseteq Y \,:\, Y' \in \mathcal{N}^0} \chi_A(Y'), \qquad A \in \mathfrak{N}^0$$
$$\left(K_P(A) = \int_{\mathcal{M}} P(dX) \sum_{X' \subseteq X \,:\, X' \in \mathcal{M}^0} \chi_A(X'), \qquad A \in \mathfrak{M}^0 \right) \tag{2.6}$$

Definition 2.2. The measure K_P is called the correlation measure of the c state (resp., state) P. The restriction of K_P to $(N_n, \mathscr{C}_n)$ [resp., $(M_n, \mathscr{D}_n)$], $n = 1, 2, \ldots$, is denoted by $K_P^{(n)}$ and is called the n-particle correlation measure of P. If K_P is absolutely continuous with respect to μ (resp., λ) then the Radon–Nicodym derivative $k_P = dK_P/d\mu$ $(k_P = dK_P/d\lambda)$ is called the correlation function of P; the restriction of k_P to N_n (M_n) is denoted by $k_P^{(n)}$ and is called the n-particle correlation function.

Note that if a c state (state) P is translation invariant, then its correlation measure K_P is also translation invariant:

$$K_P(S_x A) = K_P(A), \qquad A \in \mathfrak{N}^0(\mathfrak{M}^0), \quad x \in \mathbb{R}^\nu$$

In particular the 1-particle correlation measure $K_P^{(1)}$ of a translation-invariant c state (state) P has the form

$$K_P^{(1)} = \alpha_P m \qquad \left[K_P^{(1)} = \alpha_P \left(m \times \bar{K}_P^{(1)} \right) \right] \tag{2.7}$$

where the constant $\alpha_P \in \mathbb{R}_+^1$ is called the (mean) particle density in the c state (state) P (and $\bar{K}_P^{(1)}$ is a probability measure on $\mathbb{R}^\nu$ which describes the 1-particle velocity distribution in the state P). If, in the case of states, $K_P^{(1)}$ is absolutely continuous with respect to l, then the 1-particle correlation function $k_P^{(1)}$ admits the representation

$$k_P^{(1)}(q, v) = \alpha_P \bar{k}_P^{(1)}(v), \qquad (q, v) \in M \tag{2.8}$$

where $\bar{k}_P^{(1)} = d\bar{K}_P^{(1)}/dm$.

In the general (non-translation-invariant) case the 1-particle correlation function $k_P^{(1)}$ of a state P (if it exists) may be represented in the form

$$k_P^{(1)}(q, v) = \alpha_P(q) \bar{k}_P^{(1)}(q, v), \qquad (q, v) \in M \tag{2.8'}$$

where α_P is a function $\mathbb{R}^1 \to \mathbb{R}_+^1$ and $\bar{k}_P^{(1)}: M \to \bar{\mathbb{R}}_+^1$ is such that $\int_{\mathbb{R}^1} k_P^{(1)}(q, v)\, dv = 1$. The value $\alpha_P(q)$ and the measure with density (with respect to dv) $\bar{k}_P^{(1)}(q, \cdot)$ are interpreted, as above, as the particle density and the 1-particle velocity distribution at the point q, respectively.

Let P be a c state (resp., state) and $C \subset \mathbb{R}^\nu$ a bounded set. The restriction of P to $\mathfrak{N}_C$ (resp., $\mathfrak{M}_C$) induces, via the map π_C, a probability measure on $[\mathcal{N}(C), \mathfrak{N}(C)]$ $[\mathcal{M}(C), \mathfrak{M}(C)]$ which is denoted by P_C and is called the local distribution of P in the volume C. A c state (resp., state) P is said to be locally absolutely continuous with respect to μ (λ) if, for any bounded $C \subset \mathbb{R}^\nu$, P_C is absolutely continuous with respect to μ (λ). It is not hard to see that if P is a locally absolutely continuous c state (resp., state), then K_p is absolutely continuous with respect to μ (resp., λ) and the correlation function k_P is given μ-a.e. (λ-a.e.) by

$$k_P(Y) = \int_{\mathcal{N}(C)} dY'\, p_P^{(C)}(Y \cup Y') \qquad \left[k_P(X) = \int_{\mathcal{M}(C)} dX'\, p_P^{(C)}(X \cup X')\right]$$

where $C \subset \mathbb{R}^\nu$ is a bounded set, $Y \in \mathcal{N}(C)$ $[X \in \mathcal{M}(C)]$ and $p_P^{(C)}$ denotes the density of P_C with respect to μ (λ).

Let P be an arbitrary c state (resp., state). Since $\mathcal{N}$ (resp., $\mathcal{M}$) is a polish space, for any σ algebra $\mathfrak{N}' \subset \mathfrak{N}$ ($\mathfrak{M}' \subset \mathfrak{M}$) the conditional probability $P(\cdot \mid \mathfrak{N}')$ $[P(\cdot \mid \mathfrak{M}')]$ is regular (see, e.g., Ref. 11, Chap. 1, Section 3, Theorem 3), i.e., it can be considered as a family of c states (states) depending on $Y \in \mathcal{N}$ ($X \in \mathcal{M}$). For a given $Y \in \mathcal{N}$ (resp., $X \in \mathcal{M}$) we denote the corresponding c state (state) by $P_{\mathfrak{N}'}(\cdot \mid Y)$ ($P_{\mathfrak{M}'}(\cdot \mid X)$). In the case $\mathfrak{N}' = \mathfrak{N}_{C^c}$ (resp., $\mathfrak{M}' = \mathfrak{M}_{C^c \times \mathbb{R}^\nu}$) where $C \subset \mathbb{R}^\nu$ is a bounded set, the local distribution $P_{\mathfrak{N}'}(\cdot \mid Y)_C$ ($P_{\mathfrak{M}'}(\cdot \mid X)_C$) is denoted by $P_C(\cdot \mid Y)$ ($P_C(\cdot \mid X)$). Such measures play a fundamental role in the definition of Gibbs states, which we now briefly recall.

Definition 2.3. A configuration potential, or, shortly, a c potential (resp., a phase potential, or, shortly, a potential) is a measurable function, $\Psi: \mathcal{N}^0 \to \overline{\mathbb{R}}^1$ ($\Phi: \mathcal{M}^0 \to \overline{\mathbb{R}}^1$) such that $\Psi(\Theta) = 0$ ($\Phi(\Theta) = 0$). Given a c potential Ψ (resp., potential Φ), we denote by $\Psi^{(n)}$ ($\Phi^{(n)}$) its restriction to N_n (M_n), $n > 1$. To any c potential Ψ (resp., potential Φ) we associate the energy $H^\Psi: \mathcal{N}^0 \to \overline{\mathbb{R}}^1$ ($H^\Phi: \mathcal{M}^0 \to \overline{\mathbb{R}}^1$) given by

$$\begin{gathered} H^\Psi(Y) = \sum_{Y' \subseteq Y} \Psi(Y'), \qquad Y \in \mathcal{N}^0 \\ \left[H^\Phi(X) = \sum_{X' \subseteq X} \Phi(X'), \qquad X \in \mathcal{M}^0\right] \end{gathered} \tag{2.9}$$

Let $C \subset \mathbb{R}^\nu$ be a bounded set. The conditional energy of a configuration $Y^{(1)} \in \mathcal{N}(C)$ [resp., realization $X^{(1)} \in \mathcal{M}(C)$] with the external condition $Y^{(2)} \in \mathcal{N}(C^c)$ $[X^{(2)} \in \mathcal{M}(C^c)]$ is defined as the limit

$$\begin{gathered} H^\Psi(Y^{(1)} \mid Y^{(2)}) = \lim_{s \to \infty} H^\Psi(Y^{(1)} \mid Y^{(2)}_{I^{(s)}}) \\ \left(H^\Phi(X^{(1)} \mid X^{(2)}) = \lim_{s \to \infty} H^\Phi(X^{(1)} \mid X^{(2)}_{I^{(s)}})\right) \end{gathered} \tag{2.10}$$

where $I^{(s)}$ denotes the cube $\{x = (x^1, \ldots, x^\nu) \in \mathbb{R}^\nu : |x^j| < s,\ 1 < j < \nu\}$ and, for any pair of finite configurations $Y_0, Y_0' \in \mathcal{N}^0$ (realizations X_0, X_0'

$\in \mathscr{M}^0$),

$$H^{\Psi}(Y_0 \mid Y_0') = \sum_{\tilde{Y} \subseteq Y_0 \cup Y_0' \,:\, \tilde{Y} \cap Y_0 \neq \varnothing} \Psi(\tilde{Y})$$
$$\left(H^{\Phi}(X_0 \mid X_0') = \sum_{\tilde{X} \subseteq X_0 \cup X_0' \,:\, \tilde{X} \cap X_0 \neq \varnothing} \Phi(\tilde{X}) \right) \tag{2.10'}$$

Denote by $\mathscr{N}_C^{\Psi}$ (resp., $\mathscr{M}_C^{\Phi}$) the set of all $Y \in \mathscr{N}$ ($X \in \mathscr{M}$) such that the conditional energy $H^{\Psi}(Y_C \mid Y_{C^c})$ $[H^{\Phi}(X_{C\times\mathbb{R}^\nu} \mid X_{C^c\times\mathbb{R}^\nu})]$ exists and is finite.

Definition 2.4. We say that P is a Gibbs c state (resp., Gibbs state) with c potential Ψ (potential Φ) if for any bounded $C \subset \mathbb{R}^\nu$,

(i) $P(\mathscr{N}_C^{\Psi}) = 1$ $(P(\mathscr{M}_C^{\Phi}) = 1)$,

(ii) for P-a.a. $Y \in \mathscr{N}$ ($X \in \mathscr{M}$) the following integral exists:

$$\Xi_C^{\Psi}(Y) = \int_{\mathscr{N}(C)} dY' \exp\left[-H^{\Psi}(Y' \mid Y_{C^c}) \right]$$
$$\left(\Xi_C^{\Phi}(X) = \int_{\mathscr{M}(C)} dX' \exp\left[-H^{\Phi}(X' \mid X_{C^c\times\mathbb{R}^\nu}) \right] \right) \tag{2.11}$$

(iii) for P-a.a. $Y \in \mathscr{N}$ ($X \in \mathscr{M}$) the measure $P_C(\cdot \mid Y)$ $[P_C(\cdot \mid X)]$ coincides with the probability measure $G^{(\Psi,C)}(\cdot \mid Y_{C^c})$ $[G^{(\Phi,C)}(\cdot \mid X_{C^c\times\mathbb{R}^\nu})]$ given by

$$G^{(\Psi,C)}(A \mid Y_{C^c}) = \Xi_C^{\Psi}(Y)^{-1} \int_A dY' \exp\left[-H^{\Psi}(Y' \mid Y_{C^c}) \right], \qquad A \in \mathfrak{N}(C)$$
$$\left(G^{(\Phi,C)}(A \mid X_{C^c\times\mathbb{R}^\nu}) = \Xi_C^{\Phi}(X)^{-1} \int_A dX' \times \exp\left[-H^{\Phi}(X' \mid X_{C^c\times\mathbb{R}^\nu}) \right], \qquad A \in \mathfrak{M}(C) \right) \tag{2.12}$$

From the definition of the Gibbs c state (resp., state) it follows immediately that P is locally absolutely continuous and hence, the correlation function k_P exists.

There is a convenient formula for the correlation function k_P of a Gibbs c state (resp., state) P:

$$k_P(Y) = \exp\left[-H^{\Psi}(Y) \right] \hat{k}_P(Y), \qquad Y \in \mathscr{N}^0$$
$$\left(k_P(X) = \exp\left[-H^{\Phi}(X) \right] \hat{k}_P(X), \qquad X \in \mathscr{M}^0 \right) \tag{2.13}$$

where

$$\hat{k}_P(Y) = \int_{\mathscr{N}} P(d\tilde{Y}) \exp\left[-W^{\Psi}(Y, \tilde{Y}) \right]$$
$$\left(\hat{k}_P(X) = \int_{\mathscr{M}} P(d\tilde{X}) \exp\left[-W^{\Phi}(X, \tilde{X}) \right] \right)$$

and $W^{\Psi}(Y, \tilde{Y}) = H^{\Psi}(Y \mid \tilde{Y}) - H^{\Psi}(Y)$ ($W^{\Phi}(X, \tilde{X}) = H^{\Phi}(X \mid \tilde{X}) - H^{\Phi}(X)$).

The measure $G^{(\Psi,C)}(\cdot \mid Y)$, $Y \in \mathscr{N}(C^c)$ (resp., $G^{(\Phi,C)}(\cdot \mid X)$, $X \in \mathscr{M}(C^c)$) is called the conditional Gibbs measure in the volume C with the c-potential Ψ (potential Φ) and the external condition Y (X). It will be convenient to introduce the c state (resp., state) [i.e., the probability measure on $(\mathscr{N}, \mathfrak{N})$ $((\mathscr{M}, \mathfrak{M}))$] which is concentrated on the set $\{\tilde{Y} \in \mathscr{N} : \tilde{Y}_{C^c} = Y\}$ $(\{\tilde{X} \in \mathscr{M} : \tilde{X}_{C^c \times \mathbb{R}^\nu} = X\})$ and whose local distribution in the volume C is $G^{(\Psi,C)}(\cdot \mid Y)$ $(G^{(\Phi,C)}(\cdot \mid X))$. This state is called the conditional Gibbs c state (resp., state) in the volume C with the c potential Ψ (potential Φ) and the condition Y (X). We denote it again by $G^{(\Psi,C)}(\cdot \mid Y)$ (resp., $G^{(\Phi,C)}(\cdot \mid X)$). The Radon–Nikodym derivative with respect to the Lebesgue measure μ (resp. λ) of the restriction of the correlation measure $K_{G^{(\Psi,C)}(\cdot \mid Y)}$ to $\mathscr{N}(C)$ $(K_{G^{(\Phi,C)}(\cdot \mid X)}$ to $\mathscr{M}(C))$ is denoted, as above, by $k_{G^{(\Psi,C)}(\cdot \mid Y)}$ $(k_{G^{(\Phi,C)}(\cdot \mid X)})$. In analogy with Eq. (2.13) one can write

$$
\begin{aligned}
&k_{G^{(\Psi,C)}(\cdot \mid \tilde{Y})}(Y) = \exp\left[-H^{\Psi}(Y)\right] \hat{k}_{G^{(\Psi,C)}(\cdot \mid \tilde{Y})}(Y), \qquad Y \in \mathscr{N}^0 \\
&\left(k_{G^{(\Phi,C)}(\cdot \mid \tilde{X})}(X) = \exp\left[-H^{\Phi}(X)\right] \hat{k}_{G^{(\Phi,C)}(\cdot \mid \tilde{X})}(X), \qquad X \in \mathscr{M}^0\right)
\end{aligned}
\tag{2.14}
$$

In the second part of this section we expose some basic facts about hard rod dynamics. One of the main points is that it can be given in terms of the free dynamics, which we now define.

Let us denote by $\mathscr{M}'$ the set of realizations $X \in \mathscr{M}$ such that, for any $\tau \in \mathbb{R}^1$, $T_\tau^0 X \in \mathscr{M}$ where

$$T_\tau^0 X = \{(q, v) \in M : (q - \tau v, v) \in X\} \tag{2.15}$$

One can check that $\mathscr{M}' \in \mathfrak{M}$, and $\{T_\tau^0, \tau \in R^1\}$ is a 1-parameter group of measurable transformations of $\mathscr{M}'$ into itself.

Definition 2.5. The transformation group $\{T_\tau^0, \tau \in \mathbb{R}^1\}$ is called the free particle dynamics.

If a state P satisfies the condition $P(\mathscr{M}') = 1$, then the formula

$$P_\tau^0(A) = P\left(T_{-\tau}^0(A \cap \mathscr{M}')\right), \qquad A \in \mathfrak{M}, \quad \tau \in \mathbb{R}^1$$

defines a family of states $\{P_\tau^0, \tau \in \mathbb{R}^1\}$ which is called the free evolution of the initial state P $(= P_0^0)$.

From now on we assume $\nu = 1$, and fix once and for all a positive number d, which will be the length of the hard rods.

It is convenient to give a representation of the configuration $Y \in \mathscr{N}$ (resp., $X \in \mathscr{M}$), which is based on the particle order. We introduce in the one-particle phase space $M = \mathbb{R}^1 \times \mathbb{R}^1$ the relation of lexicographic order by setting $(q, v) > (q', v')$ if either $q > q'$ or $q = q'$ and $v > v'$. Let $a \in \mathbb{R}^1$ be fixed. For any $Y \in \mathscr{N}$ (resp., $X \in \mathscr{M}$) we set $n_+^{(a)}(Y) = |Y_{[a,+\infty)}|$ $(n_+^{(a)}(X) = |X_{[a,+\infty) \times \mathbb{R}^1}|)$ and $n_-^{(a)}(Y) = |Y_{(-\infty,a)}|$ $(n_-^{(a)}(X) = |X_{(-\infty,a) \times \mathbb{R}^1}|)$. If $Y \neq \Theta$

(resp., $X \neq \Theta$) we can write $Y = \{q_j^{(a)}(Y),\ -[n_-^{(a)}(Y)+1] < j < n_+^{(a)}(Y)\}$ ($X = \{(q_j^{(a)}(X), v_j^{(a)}(X)),\ -[n_-^{(a)}(X)+1] < j < n_+^{(a)}(X)\}$), where, if $n_+^{(a)} > 0$, $q_0^{(a)}(Y)$ $((q_0^{(a)}(X), v_0^{(a)}(X)))$ is the lower bound of the set $Y_{[a,+\infty)}$ $(X_{[a,+\infty)\times\mathbb{R}^1})$, if $n_-^{(a)} > 0$, $q_{-1}^{(a)}(Y)$ $((q_{-1}^{(a)}(X), v_{-1}^{(a)}(X)))$ is the upper bound of the set $Y_{(-\infty,a)}$ $(X_{(-\infty,a)\times\mathbb{R}^1})$, and $q_{j+1}^{(a)}(Y) > q_j^{(a)}(Y)$ $((q_{j+1}^{(a)}(X), v_{j+1}^{(a)}(X)) > (q_j^{(a)}(X), v_j^{(a)}(X)))$ for $-(n_-^{(a)}+1) < j < n_+^{(a)} - 1$. Clearly such a representation is unique. We shall call it the canonical representation of Y (resp., X) with origin in a. If $a = 0$ we shall drop the superscript (a) in the notation.

Consider the sets

$$\tilde{\mathscr{M}}_d^+ = \{X \in \mathscr{M} : (q_{j+1}(X) - d, v_{j+1}(X)) > (q_j(X), v_j(X)),\ -(n_- + 1) < j < n_+ - 1\} \tag{2.16}$$

$$\overline{\mathscr{M}}_d = \{X \in \mathscr{M} : q_{j+1}(X) > q_j(X) + d,\ -(n_- + 1) < j < n_+ - 1\} \tag{2.17}$$

$$\mathscr{M}_d^+ = \left\{X \in \tilde{\mathscr{M}}_d^+ : \sum_{j=0}^{\infty} [q_{j+1}(X) - q_j(X) - d] = +\infty \text{ if } n_+ = +\infty \text{ and } \sum_{j=-\infty}^{-1} [q_j(X) - q_{j-1}(X) - d] = +\infty \text{ if } n_- = -\infty\right\} \tag{2.18}$$

Clearly $\tilde{\mathscr{M}}_d^+, \overline{\mathscr{M}}_d, \mathscr{M}_d^+ \in \mathfrak{M}$. We shall call $\tilde{\mathscr{M}}_d^+$ the phase space of a locally finite hard rod system. Note that colliding particles are prescribed to be in the outgoing position. For later purposes we introduce also the set

$$\mathscr{N}_d = \{Y \in \mathscr{N} : q_{j+1}(Y) > q_j(Y) + d,\ -(n_- + 1) < j < n_+ - 1\} \tag{2.19}$$

which we shall call the configuration space of a locally finite hard rod system.

For a given $C \subset \mathbb{R}^1$ we introduce, in analogy with (2.16)–(2.19), the sets $\tilde{\mathscr{M}}_d^+(C)$, etc., which are the intersections of the previous sets with $\mathscr{M}(C)$ or $\mathscr{N}(C)$.

We describe now how hard rod dynamics can be given in terms of free dynamics.

For any $a \in \mathbb{R}^1$ consider the map $D_a : \mathscr{N} \to \mathscr{N}_d$ (resp., $\mathscr{M} \to \mathscr{M}_d^+$) defined by

$$\begin{aligned} &q_j^a(D_a Y) = q_j^a(Y) + jd, \qquad Y \in \mathscr{N} \\ &(q_j^a(D_a X) = q_j^a(X) + jd, \qquad v_j^a(D_a X) = v_j(X), \qquad X \in \mathscr{M}), \\ &\qquad\qquad -(n_- + 1) < j < n_+ \end{aligned} \tag{2.20}$$

If $\mathcal{N}_d^{(a)}$ (resp., $\mathcal{M}_d^{(a)}$) denotes the image of $\mathcal{N}$ ($\mathcal{M}$) under D_a it is easily seen that $\mathcal{N}_d^{(a)} = \{Y \in \mathcal{N}_d : Y_{[a-d,a)} = \Theta\}$ ($\mathcal{M}_d^{(a)} = \{X \in \mathcal{M}_d^+ : X_{[a-d,a)\times\mathbb{R}^1} = \theta\}$). The map D_a may be considered as a "dilatation" of the configurations (resp., realizations) around the point a.

The inverse map for D_a is defined on $\mathcal{N}_d^{(a)}$ (resp., $\mathcal{M}_d^{(a)}$) and denoted by C_a. It is easily seen that

$$\begin{aligned} &q_j^a(C_aY) = q_j^a(Y) - jd, \qquad Y \in \mathcal{N}_d^{(a)} \\ &\big(q_j^a(C_aX) = q_j^a(X) - jd, \qquad v_j^a(C_aX) = v_j(X), \qquad X \in \mathcal{M}_d^{(a)}\big), \\ &\qquad\qquad -(n_- + 1) < j < n_+ \end{aligned} \tag{2.21}$$

The map C_a can be considered as a "contraction" of the configurations (resp., realizations) to the point a.

For any $X \in \mathcal{M}$ and $(q,v) \in X$ we set

$$\begin{aligned} X_{(q,v)}^L(\tau) &= \{(q',v') \in X : (q',v') < (q,v), (q+\tau v, v) < (q'+\tau v', v')\} \\ X_{(q,v)}^R(\tau) &= \{(q',v') \in X : (q',v') > (q,v), (q+\tau v, v) > (q'+\tau v', v')\}, \\ &\qquad \tau \in \mathbb{R}^1 \end{aligned} \tag{2.22}$$

$X_{(q,v)}^L(\tau)$ and $X_{(q,v)}^R(\tau)$ are the "subrealizations" of X consisting of all the particles whose trajectories intersect the trajectory of (q,v) from right to left and from left to right, respectively, in the course of the free motion up to the time τ.

Let $\mathcal{M}'' \subset \mathcal{M}$ be the set of all $X \in \mathcal{M}$ such that $|X_{(q,v)}^L(\tau)| < +\infty$ and $|X_{(q,v)}^R(\tau)| < +\infty$ for any $\tau \in \mathbb{R}^1$ and $(q,v) \in M$. Consider further the set $\mathcal{M}_d'$ defined by

$$\mathcal{M}_d' = \{X \in \mathcal{M}_d^+ : C_qX \in \mathcal{M}'' \text{ for any } (q,v) \in X\} \tag{2.23}$$

Clearly, both $\mathcal{M}'', \mathcal{M}_d' \in \mathfrak{M}$. Given $X \in \mathcal{M}_d'$, $(q,v) \in M$ and $\tau \in \mathbb{R}^1$, we set

$$w_{X,v,\tau}(q) = \bar{q} + \tau v + d n_X(q,v,\tau) \tag{2.24}$$

where

$$\bar{q} = \bar{q}(q,X) = \begin{cases} q, & \text{if } X_{[q-d,q)} = \Theta \\ \tilde{q}, & \text{if } |X_{[q-d,q)\times\mathbb{R}^1}| = 1,\ X_{[q-d,q)\times\mathbb{R}^1} = \{(\tilde{q},\tilde{v})\} \text{ and } \tilde{v} > v \\ \tilde{q} + d, & \text{if } |X_{[q-d,q)\times\mathbb{R}^1}| = 1,\ X_{[q-d,q)\times\mathbb{R}^1} = \{(\tilde{q},\tilde{v})\} \text{ and } \tilde{v} < v \end{cases} \tag{2.25}$$

$$n_X(q,v,\tau) = n_X^+(q,v,\tau) - n_X^-(q,v,\tau) \tag{2.26}$$

and

$$n_X^+(q,v,\tau) = |(C_{\bar{q}}X)^R_{(\bar{q},v)}(\tau)|, \qquad n_X^-(q,v,\tau) = |(C_{\bar{q}}X)^L_{(\bar{q},v)}(\tau)| \quad (2.27)$$

The meaning of the objects we introduced is the following. If $(q,v) \in X$ the quantity $n_X(q,v,\tau)$ gives the algebraic number of collisions of the particle (q,v) in the course of the motion up to time τ. In the general case it gives the algebraic number of collisions of the "(q,v)-test particle" with the "real" particles up to time t. By a test particle we mean here a pointlike particle which moves inertially between collisions, jumps of $\pm d$ when it collides with a real particle, without affecting the motion of the real particle, and at collision is prescribed to be in the outgoing position.

Thus, for given $X \in \mathscr{M}_d'$, $v,\tau \in \mathbb{R}^1$ we can consider $w_{X,y,\tau}$ as a map $\mathbb{R}^1 \to \mathbb{R}^1$. Given $X \in \mathscr{M}_d'$, the family of maps $w_{X,v,\tau}$, $v \in \mathbb{R}^1$, $\tau \in \mathbb{R}^1_+$, has the following monotonicity property: if $(q',v') \in X$, $q' < q$ and $v' < v$, then $w_{X,v',\tau}(q') \dot{<} w_{X,v,\tau}(q)$ for all $\tau \in \mathbb{R}^1_+$. Likewise, if $(q'',v'') \in X$, $q'' > q$ and $v' > v$, then $w_{X,v'',\tau}(q'') > w_{X,v,\tau}(q)$ for all $\tau \in \mathbb{R}^1_+$.

Definition 2.6. The hard rod dynamics, $\{T_\tau^d, \tau \in \mathbb{R}^1\}$, is defined on $\mathscr{M}_d'$ by

$$T_\tau^d X = \{(q,v) \in M : q = w_{X,v,\tau}(\tilde{q}) \text{ for some } (\tilde{q},v) \in X\} \quad (2.28)$$

The motion of a single hard is given by

$$(q,v) \in X \to [w_{X,v,\tau}(q), v], \qquad \tau \in \mathbb{R}^1 \quad (2.29)$$

The following assertion immediately follows from the arguments developed in Ref. 3 (see Ref. 3, Section 7, Propositions 7.2 and 7.3):

Lemma 2.1. For any $X \in \mathscr{M}_d'$, $\tau \in \mathbb{R}^1$, and $(q_0, v_0) \in X$ the realization $T_\tau^d X$ is given by

$$T_\tau^d X = S_b D_{q_0+\tau v_0} T_\tau^0 C_{q_0} X \quad (2.30)$$

where

$$b = d \cdot n_X(q_0, v_0, \tau) \quad (2.31)$$

Furthermore, $T_\tau^a X \in \mathscr{M}_d'$ and the family $\{T_\tau^d, \tau \in \mathbb{R}^1\}$ is a one-parameter group of one-to-one transformations of $\mathscr{M}_d'$ onto $\mathscr{M}_d'$.

Definition 2.7. Given a state P such that $P(\mathscr{M}_d') = 1$, we define the hard rod time evolution $\{P_\tau, \tau \in \mathbb{R}^1\}$ by

$$P_\tau(A) = P(T^d_{-\tau}(A \cap \mathscr{M}_d')), \qquad A \in \mathfrak{M}, \quad \tau \in \mathbb{R}^1 \quad (2.32)$$

3. AN EXISTENCE AND UNIQUENESS THEOREM FOR THE LIMIT EQUATION

This section is devoted to a theorem of existence and uniqueness for the solution of Eq. (1.1). We study here the following Cauchy problem:

$$\frac{\partial}{\partial t} g(q,v,t) + v\frac{\partial}{\partial q} g(q,v,t) + d\frac{\partial}{\partial q}\left[g(q,v,t)\left[1 - d\int_{\mathbb{R}^1} dv''\, g(q,v'',t)\right]^{-1} \right.$$

$$\left. \times \int_{\mathbb{R}^1} dv'\,(v-v')g(q,v',t)\right] = 0 \qquad (q,v)\in M, \quad t\in\mathbb{R}^1 \tag{3.1}$$

with the initial data

$$g(q,v,0) = g_0(q,v), \qquad (q,v)\in M \tag{3.2}$$

Given a function f on M and $a\in\mathbb{R}^1$ we denote by $S_a f$ the translated function:

$$S_a f(q,v) = f(q-a,v), \qquad (q,v)\in M, \quad a\in\mathbb{R}^1$$

Definition 3.1. We denote by $\mathscr{L}$ the class of nonnegative measurable functions $\varphi:\mathbb{R}^1\to\mathbb{R}^1_+$ such that

$$\int_{\mathbb{R}^1} dv\,\varphi(v) < \infty, \qquad \int_{\mathbb{R}^1} dv\,|v|\varphi(v) < \infty$$

Definition 3.2. Let $\psi\in\mathscr{L}$. We denote by $\mathscr{F}_\psi$ the class of nonnegative measurable functions $f:\mathbb{R}^1\to\mathbb{R}^1_+$ such that

(i) for any fixed $v\in\mathbb{R}^1$ the function $f(\cdot,v)\in C^1$,

(ii) for any $(q,v)\in M$

$$\max\left[f(q,v), |(\partial/\partial q)f(q,v)|\right] \leqslant \psi(v)$$

Denote by $\mathscr{F}$ the union $\bigcup_{\psi\in\mathscr{L}}\mathscr{F}_\psi$.

Given $f\in\mathscr{F}$, we set

$$\sigma_f(q) = \int_{\mathbb{R}^1} dv\, f(q,v), \qquad \zeta_f(q) = \int_{\mathbb{R}^1} dv\, vf(q,v), \qquad q\in\mathbb{R}^1 \tag{3.3}$$

Using the dominated convergence theorem it can be seen that the functions $\sigma_f:\mathbb{R}^1\to\mathbb{R}^1_+$ and $\zeta_f:\mathbb{R}^1\to\mathbb{R}^1$ are of class C^1. If $f\in\mathscr{F}_\varphi$, then for any $q\in\mathbb{R}^1$

$$\max\left[\sigma_f(q), |(d/dq)\sigma_f(q)|\right] \leqslant \int_{\mathbb{R}^1} dv\,\varphi(v) \tag{3.4a}$$

$$\max\left[|\zeta_f(q)|, |(d/dq)\zeta_f(q)|\right] \leqslant \int_{\mathbb{R}^1} dv\,|v|\varphi(v) \tag{3.4b}$$

In what follows we set $\bar\sigma_f = \sup_{q\in\mathbb{R}^1}\sigma_f(q)$.

Definition 3.3. Let $f \in \mathscr{F}$, $q_0 \in \mathbb{R}^1$. We denote by A_{f,q_0} the map $\mathbb{R}^1 \to \mathbb{R}^1$ given by

$$A_{f,q_0}(q) = q + d\int_{q_0}^{q} dq'\, \sigma_f(q'), \qquad q \in \mathbb{R}^1 \tag{3.5}$$

It is easily seen that if $f \in \mathscr{F}_\varphi$, then for any $q_0 \in \mathbb{R}^1$

$$1 \leqslant \frac{d}{dq} A_{f,q_0}(q) \leqslant 1 + d\int_{\mathbb{R}^1} dv\,(v), \qquad q \in \mathbb{R}^1$$

and hence, A_{f,q_0} is a diffeomorphism of $\mathbb{R}^1$ onto itself.

Definition 3.4. Let $f \in \mathscr{F}$, $q_0 \in \mathbb{R}^1$. We set

$$(\mathbf{A}_{q_0} f)(q,v) = f\big((A_{f,q_0})^{-1}(q), v\big)\Big[1 + d\sigma_f\big((A_{f,q_0})^{-1}(q)\big)\Big]^{-1}, \qquad (q,v) \in M \tag{3.6}$$

Definition 3.5. Let $\psi \in \mathscr{F}$, $r \in [0, d^{-1})$, and let $\mathscr{G}_{\psi,r}$ denote the class of functions $g \in \mathscr{F}_\psi$ with $\bar{\sigma}_g \leqslant r$. We denote by $\mathscr{G}_\psi$ the union $\bigcup_{r\in[0,d^{-1})} \mathscr{G}_{\psi,r}$, and by $\mathscr{G}$ the union $\bigcup_{\psi\in\mathscr{L}} \mathscr{G}_\psi$.

Proposition 3.I. For any $f \in \mathscr{F}$ and $q_0 \in \mathbb{R}^1$ the function $\mathbf{A}_{q_0} f$ is in $\mathscr{G}$. Hence, $\mathbf{A}_{q_0}$ is a map $\mathscr{F} \to \mathscr{G}$.

Proof. Let $f \in \mathscr{F}_\psi$. Setting $g = \mathbf{A}_{q_0} f$ and $\tilde{q} = A_{f,q_0}(q)$, $q \in \mathbb{R}^1$, we have

$$\sigma_g(q) = \sigma_f(\tilde{q})(1 + d\sigma_f(\tilde{q}))^{-1} \leqslant \int_{\mathbb{R}^1} dv\, \psi(v)\Big[1 + d\int_{\mathbb{R}^1} dv'\, \psi(v')\Big]^{-1}, \qquad q \in \mathbb{R}^1 \tag{3.7}$$

Hence, $g \in \mathscr{G}_\varphi$ with $\varphi(v) = \psi(v)[1 + d\int_{\mathbb{R}^1} dv'\,\psi(v')]^{-1}$, $v \in \mathbb{R}^1$.

Definition 3.6. Let $g \in \mathscr{G}$, $q_0 \in \mathbb{R}^1$. We denote by B_{g,q_0} the map $\mathbb{R}^1 \to \mathbb{R}^1$ given by

$$B_{g,q_0}(q) = q - d\int_{q_0}^{q} dq'\, \sigma_g(q'), \qquad q \in \mathbb{R}^1 \tag{3.8}$$

One can see that for any $g \in \mathscr{G}$ and $q_0 \in \mathbb{R}^1$

$$0 < 1 - d\bar{\sigma}_g \leqslant (d/dq) B_{g,q_0}(q) \leqslant 1, \qquad q \in \mathbb{R}^1$$

and hence, B_{g,q_0} is a diffeomorphism of $\mathbb{R}^1$ onto itself.

Definition 3.7. Let $g \in \mathscr{G}$, $q_0 \in \mathbb{R}^1$. We set

$$(\mathbf{B}_{q_0} g)(q,v) = g\big((B_{g,q_0})^{-1}(q), v\big)\Big[1 - d\sigma_g\big((B_{g,q_0})^{-1}(q)\big)\Big]^{-1}, \qquad (q,v) \in M \tag{3.9}$$

Proposition 3.II. For any $g \in \mathscr{G}$ and $q_0 \in \mathbb{R}^1$ the function $\mathbf{B}_{q_0} g$ is in $\mathscr{F}$. Hence, $\mathbf{B}_{q_0}$ is a map $\mathscr{G} \to \mathscr{F}$.

Proof. Let $g \in \mathscr{G}_\psi$. Then $\mathbf{B}_{q_0} g \in \mathscr{F}_\varphi$ with $\varphi = \psi(1 - d\bar{\sigma}_g)^{-1}$. ■

Some comment on the objects introduced so far is in order. For the "mechanical" intuition a function $f \in \mathscr{F}$ may be considered as the density of mass of a fluid on the line $\mathbb{R}^1$. This "fluid" is actually composed of different "fluid species," labeled by the velocity v. The total mass density and the total momentum density of the fluid at the point q are given by $\sigma_f(q)$ and $\zeta_f(q)$, respectively. The map A_{f,q_0} is a dilatation of the line around q_0: each point q is shifted away from q_0, at a distance which is equal to d times the total mass of the fluid between q_0 and q. $\mathbf{A}_{q_0} f$ is the mass distribution of the stretched fluid. Intuitively dilatation is obtained by letting each fluid element acquire an additional volume which is d times its mass. Therefore $\mathbf{A}_{q_0}$ can be regarded as a transformation which for any mass distribution of a fluid of pointlike particles gives us a corresponding distribution of a fluid of hard rods with length d. The map $\mathbf{B}_{q_0}$ describes the converse operation, by which we contract the fluid.

The maps $\mathbf{A}_{q_0}$ and $\mathbf{B}_{q_0}$ can be considered as the continuum analogs of the transformations D_{q_0} and C_{q_0} given by Eqs. (2.20) and (2.21).

The main properties of the maps $\mathbf{A}_{q_0}$ and $\mathbf{B}_{q_0}$ we need below are given in the following proposition.

Proposition 3.III. (i) Let $f \in \mathscr{F}$ (resp., $g \in \mathscr{G}$), $q_0 \in \mathbb{R}^1$ and $g = \mathbf{A}_{q_0} f$ ($f = \mathbf{B}_{q_0} g$). Then $B_{g,q_0} = (A_{f,q_0})^{-1}$.

(ii) For any $q_0 \in \mathbb{R}^1$ the map $\mathbf{A}_{q_0}$ is a bijection of $\mathscr{F}$ onto $\mathscr{G}$ and $\mathbf{A}_{q_0}^{-1} = \mathbf{B}_{q_0}$.

(iii) For any $f \in \mathscr{F}$ and $a, q_0, q_1 \in \mathbb{R}^1$:

$$S_a \mathbf{A}_{q_0} f = \mathbf{A}_{q_1} S_b f \tag{3.10}$$

where

$$b = q_1 - (A_{f,q_0})^{-1}(q_1 - a) \tag{3.11}$$

(iv) For any $g \in \mathscr{G}$ and $a, q_0, q_1 \in \mathbb{R}^1$

$$S_b \mathbf{B}_{q_1} g = \mathbf{B}_{q_0} S_a g \tag{3.12}$$

where

$$b = B_{S_a g, q_0}(q_1 + a) - q_1 \tag{3.13}$$

Proof. (i) Let $f \in \mathscr{F}$, $g = \mathbf{A}_{q_0} f$. Evidently, $B_{g,q_0}(A_{f,q_0}(q_0)) = q_0$. Therefore, to prove the inequality $B_{g,q_0} = (A_{f,q_0})^{-1}$ it is enough to check that the derivative

$$(d/dq)B_{g,q_0}(A_{f,q_0}(q)) = \left[(d/dq)A_{f,q_0}(q)\right]\left[(d/d\tilde{q})B_{g,q_0}(\tilde{q})|_{\tilde{q}=A_{f,q_0}(q)}\right]$$
$$= \left[1 + d\sigma_f(q)\right]\left\{1 - d\sigma_g(A_{f,q_0}(q))\right\}$$

is identically 1. But this follows from the equality (3.7). ∎

The proof of the equality $B_{g,q_0} = (A_{f,q_0})^{-1}$ for $g \in \mathscr{G}$ and $f = \mathbf{B}_{q_0} g$ is similar.

(ii) Let again $f \in \mathscr{F}$, $g = \mathbf{A}_{q_0} f$. Let $f' = \mathbf{B}_{q_0} g$. Using Definition 3.7 and assertion (i) we have

$$f'(q,v) = g(A_{f,q_0}(q),v)\left\{1 - d\sigma_g\left[A_{f,q}(q)\right]\right\}^{-1} \tag{3.14}$$

From the equality (3.7) we find $1 - d\sigma_g(A_{f,q_0}(q)) = \{1 + d\sigma_g(A_{f,q}(q))\}^{-1}$ whence, substituting into (3.14) and using Definition 3.4, we get $f' = f$. Therefore $\mathbf{A}_{q_0} : \mathscr{F} \to \mathscr{G}$ is an injection and $B_{q_0} : \mathscr{G} \to \mathscr{F}$ is a surjection with $\mathbf{A}_{q_0}^{-1} = \mathbf{B}_{q_0}$. In a similar way we can prove that $\mathbf{B}_{q_0}$ is an injection and A_{q_0} is a surjection with $\mathbf{B}_{q_0}^{-1} = \mathbf{A}_{q_0}$.

(iii) Let $f \in \mathscr{F}$. Given $a, q_0, q_1 \in \mathbb{R}^1$, let b be defined by (3.11). Set $f_{(b)} = S_b f$. From Definition 3.4 we see that (3.10) follows from the equality

$$(A_{f,q_0})^{-1}(q - a) = (A_{f_{(b)},q_1})^{-1}(q) - b, \qquad q \in \mathbb{R}^1 \tag{3.15}$$

From Definition 3.3 it is easy to see that the right-hand side of (3.15) coincides with $q^{(b)} = (A_{f,q_1-b})^{-1}(q - b)$, and therefore (3.15) is equivalent to

$$q - a = q^{(b)} + d\int_{q_0}^{q^{(b)}} dq'\, \sigma_f(q'), \qquad q \in \mathbb{R}^1 \tag{3.16}$$

It is convenient to write the left-hand side of (3.16) in the form $(q - b) + (b - a)$ and regard both sides of (3.16) as functions of the variable $q^{(b)} \in \mathbb{R}^1$. It is evident that the derivatives of both sides in $q^{(b)}$ are the same; hence, to prove (3.16), it is enough to check it [or, equivalently, (3.15)] for $q = q_1$. But for $q = q_1$ Eq. (3.15) coincides with Eq. (3.11).

(iv) The proof of (iv) is similar to that of (iii). The only difference is that the equation

$$(B_{g,q_1})^{-1}(q - b) = (B_{g_{(a)},q_0})^{-1}(q) - a, \qquad q \in \mathbb{R}^1 \tag{3.17}$$

which is the analog of Eq. (3.15) with $g_{(a)} = S_a g$, should be checked for $q = q_1 + b$.

Our proof of the existence and uniqueness theorem will be based on an explicit construction of the evolution of the density of mass. One of the main tools is again free evolution.

Definition 3.8. Given $f \in \mathscr{F}$, its free time evolution $\{T_t^0 f,\ t \in \mathbb{R}^1\}$ is defined by

$$T_t^0 f(q,v) = f(q - vt, v), \qquad (q,v) \in M \tag{3.18}$$

It is easy to see that for any $f \in \mathscr{F}$ $T_t^0 f \in \mathscr{F}$ for all $t \in \mathbb{R}^1$.

We now construct the motion of the points (or "elements") of the hard rod fluid. Given $g \in \mathscr{G}$, $(q,v) \in M$, and $t \in \mathbb{R}^1$, we set [compare with Eqs. (2.24)–(2.27)]

$$u_{g,v,t}(q) = q + tv + dm_g(q,v,t) \tag{3.19}$$

where

$$m_g(q,v,t) = m_g^+(q,v,t) - m_g^-(q,v,t) \tag{3.20}$$

with

$$m_g^+(q,v,t) = \int_q^\infty dq' \int_{-\infty}^{v - t^{-1}(q'-q)} dv'\, f_q(q',v') \tag{3.21,+}$$

$$m_g^-(q,v,t) = \int_{-\infty}^q dq' \int_{v + t^{-1}(q-q')}^\infty dv'\, f_q(q',v') \tag{3.21,−}$$

$$f_q = \mathbf{B}_q g \tag{3.22}$$

Lemma 3.IV. For any $g \in \mathscr{G}$ and $v, t \in \mathbb{R}^1$ the map $u_{g,v,t}: \mathbb{R}^1 \to \mathbb{R}^1$ given by (3.19)–(3.22) is a diffeomorphism of $\mathbb{R}^1$ into itself. Furthermore, for any $(q_0, v_0) \in M$ the following formulas are true:

$$u_{g,v,t}(q) = B_{g,q_0}(q) + tv + dm_g(q_0, v_0, t) + d\int_{q_0 + tv_0}^{B_{g,q_0}(q) + tv} dq'\, \sigma_{T_t^0 f^0}(q'), \qquad q \in \mathbb{R}^1 \tag{3.23a}$$

and

$$(d/dq)u_{g,v,t}(q) = \left[1 - d\sigma_g(q)\right]\left(1 + d\sigma_{T_t^0 f^0}(B_{g,q_0}(q) + vt)\right), \qquad q \in \mathbb{R}^1 \tag{3.23b}$$

where $f^0 = f_{q_0} = \mathbf{B}_{q_0} g$.

Proof. Let $g \in \mathscr{G}$. It is convenient to rewrite the quantity $m_g(q,v,t)$ defined in (3.20), (3.21) in the form

$$m_t(q,v,t) = \int_{\mathbb{R}^1} dv' \int_q^{q + t(v - v')} dq'\, f_q(q',v'), \qquad (q,v) \in M \tag{3.24}$$

Using Definition 3.7 and assertion (iv) of Proposition 3.III, we obtain from (3.22) that for any $q_0 \in \mathbb{R}^1$ and all $q \in \mathbb{R}^1$

$$f_q(q',v') = f^0(q' - q + B_{g,q_0}(q), v'), \qquad (q',v') \in M$$

Hence, (3.24) is equivalent to

$$m_g(q,v,t) = \int_{\mathbb{R}^1} dv' \int_{\hat{q}}^{\hat{q}+(v-v')t} dq'\, f^0(q',v'), \qquad (q,v) \in M \quad (3.25a)$$

where $\hat{q} = B_{g,q_0}(q)$. From Definitions 3.6 and 3.7 it is easy to see that

$$\hat{q} = q - d\int_{q_0}^{\hat{q}} dq'\, \sigma_{f^0}(q') \quad (3.25b)$$

Now using (3.25a, b) it is not hard to check that the right-hand side of (3.23a) coincides with that of (3.19).

From (3.19) and (3.25) we find

$$(d/dq)u_{g,v,t}(q) = 1 + d\left[\sigma_{T_t^0 f^0}(\hat{q} + tv) - \sigma_{f^0}(\hat{q})\right] d\hat{q}/dq \quad (3.25c)$$

According to the equality (3.7) and assertion (i) of Proposition 3.III, $\sigma_{f^0}(\hat{q}) = \sigma_g(q)[1 - d\sigma_g(q)]^{-1}$. According to Definition 3.6, $d\hat{q}/dq = 1 - d\sigma_g(q)$. Hence, the right-hand side of (3.25c) coincides with the right-hand side of (3.23b). ■

Corollary 3.V. For any $g \in \mathscr{G}$ the maps $u_{g,v,t}, v, t \in \mathbb{R}^1$ have the following monotonicity property:

$$u_{g,v'',t}(q'') \geqslant u_{g,v',t}(q') \qquad \text{if} \quad q'' \geqslant q', \quad v'' \geqslant v', \quad \text{for all} \quad t \geqslant 0 \quad (3.26)$$

Proof. An easy check is one based on the representation (3.23a). ■

The application

$$(q,v) \rightarrow (u_{g,v,t}(q), v), \qquad t \in \mathbb{R}^1 \quad (3.27)$$

gives the trajectory of a point in the fluid. The motion of the fluid points can be seen as a composition of the free motion $(q,v) \rightarrow (q + vt, v)$ and a space shift $q \rightarrow q + dm_g(q,v,t)$. $m_g(q,v,t)$ is the mass of the fluid which crosses the trajectory of the fluid point (q,v) from right to left, minus the mass which crosses from left to right. A comparison with Eqs. (2.29) and (2.24)–(2.27), shows that the motion of a fluid point (3.27) is the continuum analog of the motion of a single hard rod.

If the points of the fluid of initial density g move according to Eq. (3.27), the density of mass of the fluid at time t will be given by the function $\mathscr{U}_t g$:

$$\mathscr{U}_t g(q,v) = g\left(u_{g,v,t}^{-1}(q), v\right) \frac{d}{dq} u_{g,v,t}^{-1}(q) \quad (3.28)$$

Remark. Note, for later use, that, as it follows from Eqs. (3.19), (3.24), $u_{g,v,t}$ is smooth in v and in t, as well as in q, and that from Eq. (3.23b) it follows that for all $g \in \mathscr{G}$ the same is true for $u_{g,v,t}^{-1}$.

We shall prove that the construction (3.28) does actually provide a solution of our problem in some class of functions, which we now make precise.

Definition 3.9. We denote by $\mathscr{H}$ the class of the nonnegative measurable functions $h: M \times \mathbb{R}^1 \to \mathbb{R}^1_+$ such that

(i) $h(\cdot, v, \cdot) \in C^1$ for all $v \in \mathbb{R}^1$;

(ii) $h(\cdot, \cdot, t) \in \mathscr{G}_{\psi,r}$ for some $r \in [0, d^{-1})$, some $\psi \in \mathscr{L}$ and all $t \in \mathbb{R}^1$.

We shall sometimes write $h_t(q,v)$ instead of $h(q,v,t)$ to stress that the function h is considered for a fixed value of t.

Theorem 3.VI. Let $g_0 \in \mathscr{G}$. Then in the class $\mathscr{H}$ there is a unique solution, g_t, $t \in \mathbb{R}^1$, of the Cauchy problem (3.1), (3.2). This solution is given by $g_t = \mathscr{U}_t g_0$, where $\mathscr{U}_t g_0$ is given by Eq. (3.28).

The key point in the proof of Theorem 3.VI is the following result.

Lemma 3.VII. Let $g_0 \in \mathscr{G}$. Then for any $(q_0, v_0) \in M$ and $t \in \mathbb{R}^1$

$$\mathscr{U}_t g_0 = S_a \mathbf{A}_{q_0 + tv_0} T_t^0 \mathbf{B}_{q_0} g_0 \tag{3.29}$$

where

$$a = dm_{g_0}(q, v, t) \tag{3.30}$$

Before going to the proof of Lemma 3.VII we make some comments on Eqs. (3.29), (3.30). They show that in order to get the density of mass at time t, we should fix an arbitrary point $(q_0, v_0) \in M$, contract the initial density g_0 around q_0, submit the resulting mass distribution to the free evolution up to time t, then dilate around $q_0 + tv_0$, and finally shift the result by $dm_{g_0}(q_0, v_0, t)$. The evolution of the hard rod fluid is—not surprisingly—a continuum variant of the hard rod evolution [see (2.30), (2.31)].

Proof. We set again $f^0 = \mathbf{B}_{q_0} g_0$ and $g_t = \mathscr{U}_t g_0$. Using Lemma 3.IV we get

$$\begin{aligned}(d/dq) u^{-1}_{g_0,v,t}(q) &= \left[(d/d\tilde{q}) u_{g_0,v,t}(\tilde{q})|_{\tilde{q} = u^{-1}_{g_0,v,t}(q)}\right]^{-1} \\ &= \left\{1 - d\sigma_{g_0}\left[u^{-1}_{g_0,v,t}(q)\right]\right\}^{-1}\left[1 + d\sigma_{T_t^0 f^0}(\hat{q} + vt)\right]^{-1}, \\ &\qquad (q,v) \in M \end{aligned} \tag{3.31}$$

where $\hat{q} = B_{g,q_0}[u^{-1}_{g_0,v,t}(q)]$. Notice that

$$A_{T_t^0 f^0, q_0 + tv_0}(\hat{q} + vt) = q - a, \qquad (q,v) \in M \tag{3.32}$$

where a is given by (3.30). In fact, from (3.20), (3.18) and Definition 3.3 it follows that

$$\begin{aligned} d&\left[m_{g_0}(q_0, v, t) - m_{g_0}(q_0, v_0, t)\right] \\ &= d\int_{\mathbb{R}^1} dv' \int_{q_0 + t(v_0 - v')}^{q_0 + t(v - v')} dq'\, f^0(q', v') = d\int_{q_0 + tv_0}^{q_0 + tv} dq'\, \sigma_{T_t^0 f^0}(q') \\ &= A_{T_t^0 f^0, q_0 + tv_0}(q_0 + tv) - (q_0 + tv) \end{aligned} \tag{3.33}$$

Using (3.33) we see that (3.32) is true for $q = u_{g_0, v, t}(q_0)$. Because of (3.31) the derivative

$$\begin{aligned} &(d/dq) A_{T_t^0 f^0, q_0 + v_0 t}(\hat{q} + tv) \\ &\quad = \left(1 + d\sigma_{T_t^0 f^0}(\hat{q} + tv)\right)\left\{1 - d\sigma_{g_0}\left(u_{g_0, v, t}^{-1}(q)\right)\right\}(d/dq) u_{g_0, v, t}^{-1}(q) \end{aligned}$$

is equal to 1, and equality (3.32) is proved. ∎

Writing $u_{g_0, v, t}^{-1}(q) = (B_{g_0, q_0})^{-1}(\hat{q})$ and using (3.28), (3.19), (3.31), (3.32), and Definition 3.7 we find that

$$\begin{aligned} g_t(q, v) &= g_0\left((B_{g_0, q_0})^{-1}(\hat{q}), v\right)\left\{1 - d\sigma_{g_0}\left((B_{g_0, q_0})^{-1}(\hat{q})\right)\right\}^{-1} \\ &\quad \times \left[1 + d\sigma_{T_t^0 f^0}(\hat{q} + tv)\right]^{-1} \\ &= f^0(\hat{q}, v)\left[1 + d\sigma_{T_t^0 f^0}(\hat{q} + tv)\right]^{-1} \\ &= T_t^0 f^0(\hat{q} + tv, v)\left[1 + d\sigma_{T_t^0 f^0}(\hat{q} + tv)\right]^{-1} \\ &= \mathbf{A}_{q_0 + tv_0} T_t^0 f^0\left(A_{T_t^0 f^0, q_0 + tv_0}(\hat{q} + tv), v\right) = \mathbf{A}_{q_0 + tv_0} T_t^0 f^0(q - a, v) \end{aligned}$$

which coincides with the right-hand side of (3.29).

Lemma 3.VIII. Let $g_0 \in \mathscr{G}$ and let $\mathscr{U}_t g_0$ be defined by Eq. (3.28) [or, equivalently, by Eq. (3.29)]. Then (i) the function $g(q, v, t) = \mathscr{U}_t g_0(q, v)$ belongs to $\mathscr{H}$, and (ii) the maps $\mathscr{U}_t : g_0 \in \mathscr{G} \to \mathscr{U}_t g_0 \in \mathscr{G}$, $t \in \mathbb{R}^1$, have the group property, i.e., for all $g_0 \in \mathscr{G}$ and for all $t_0, t_1 \in \mathbb{R}^1$

$$\mathscr{U}_{t_1} \mathscr{U}_{t_0} g_0 = \mathscr{U}_{t_1 + t_0} g_0 \tag{3.34}$$

Proof. The first assertion follows immediately from the representation (3.29) and Propositions 3.I, 3.II [see also the remark which follows Eq. (3.28)]. To prove (ii) we choose $(q_0, v_0) \in M$ and set $f^0 = B_{q_0} g$, $a_0 = dm_{g_0}(q_0, v_0, t)$. Furthermore we set $q_1 = q_0 + tv_0 + a_0$, $g^1 = \mathscr{U}_{t_1} g_0$, $f^1 = B_{q_1} g^1$, and $a_1 = dm_{g^1}(q_1, v_0, t_1)$. According to (3.29)

$$\mathscr{U}_{t_1} g_0 = S_{a_0} \mathbf{A}_{q_0 + t_0 v_0} T_{t_0}^0 f^0, \qquad \mathscr{U}_{t_1} \mathscr{U}_{t_0} g_0 = S_{a_1} \mathbf{A}_{q_1 + t_1 v_0} T_{t_1}^0 f^1$$

Now, owing to assertions (ii) and (iv) of Proposition 3.III,

$$f^1 = \mathbf{B}_{q_1} S_a \mathbf{A}_{q_0+t_0v_0} T_{t_0}^0 f^0 = S_a \mathbf{B}_{q_0+t_0v_0} T_{t_0}^0 f^0 = S_{a_0} T_{t_0} f^0$$

Next remark that

$$m_{g^1}(q_1, v_0, t_1) = \int_{\mathbb{R}^1} dv' \int_{q_1}^{q_1+(v_0-v')t_1} dq'\, f^1(q', v')$$

$$= \int_{\mathbb{R}^1} dv' \int_{q_0+(v_0-v')t_0}^{q_0+(v_0-v')(t_0+t_1)} dq'\, f^0(q', v')$$

and consequently

$$a_0 + a_1 = dm_g(q_0, v_0, t_0 + t_1)$$

Hence, owing to the mutual commutativity of the maps S_a and T_t^0 and assertion (iii) of Proposition 3.III,

$$\mathscr{U}_{t_1} \mathscr{U}_{t_0} g_0 = S_{a_1} \mathbf{A}_{q_1+t_1v_0} T_{t_1}^0 S_{a_0} T_{t_0}^0 f^0 = S_{a_1} \mathbf{A}_{q_0+(t_0+t_1)v_0+a_0} S_{a_0} T_{t_1+t_0}^0 f^0$$

$$= S_{a_0+a_1} \mathbf{A}_{q_0+(t_0+t_1)v_0} T_{t_1+t_0}^0 f^0$$

which, according to (3.29), coincides with $\mathscr{U}_{t_1+t_0} g_0$. ■

Now we can give the first part of the proof of Theorem 3.VI.

Proof of Theorem 3.VI. Existence. We shall show that the function $g(q, v, t) = \mathscr{U}_t g_0$, given by Eq. (3.28) is a solution of the problem (3.1), (3.2). The initial condition (3.2) is obviously satisfied. Furthermore, because of assertion (ii) of Lemma 3.VIII, in order to prove that Eq. (3.1) is satisfied by the function g for all $t \in \mathbb{R}^1$, it is enough to prove that it is satisfied at $t = 0$. We have

$$\begin{aligned}
&(\partial/\partial t)\, g(q, v, t) \\
&\quad = (\partial/\partial t)\Big[\, g_0\big(u_{g_0,v,t}^{-1}(q), v\big)(d/dq) u_{g_0,v,t}^{-1}(q)\Big] \\
&\quad = \Big[(\partial/\partial \tilde{q}) g_0(\tilde{q}, v)|_{\tilde{q}=u_{g_0,v,t}^{-1}(q)}\Big]\Big[(\partial/\partial t) u_{g_0,v,t}^{-1}(q)\Big](d/dq) u_{g_0,v,t}^{-1}(q) \\
&\qquad + g_0\Big[u_{g_0,v,t}^{-1}(q), v\Big](\partial^2/\partial t\, \partial q) u_{g_0,v,t}^{-1}(q)
\end{aligned}$$

From (3.19)–(3.22) we find that

$$(\partial/\partial t) u_{g_0,v,t}^{-1}(q)|_{t=0} = -v - d\int_{\mathbb{R}^1} dv'\,(v - v') g_0(q, v')\big[1 - d\sigma_{g_0}(q)\big]^{-1}$$

$$(\partial/\partial q) u_{g_0,v,0}^{-1}(q) = 1$$

$$(\partial^2/\partial t\, \partial q) u_{g_0,v,t}^{-1}(q)|_{t=0} = -d(\partial/\partial q)\int_{\mathbb{R}^1} dv'\,(v - v') g_0(q, v')\big[1 - d\sigma_{g_0}(q)\big]^{-1}$$

so that Eq. (3.1) is satisfied. Furthermore $g \in \mathscr{H}$ in force of Lemma 3.VIII. Existence is proved. ■

Proof of Theorem 3.VI. Uniqueness. Given a function $h \in \mathscr{H}$, we set

$$v_h^*(q,v,t) = v + d\big[1 - d\sigma_{h_t}(q)\big]^{-1}\int_{\mathbb{R}^1} dv'(v-v')h_t(q,v'),$$

$$(q,v) \in M, \quad t \in \mathbb{R}^1 \qquad (3.35)$$

and consider the following Cauchy problem:

$$\begin{cases} \dot{q}^*(t) = v_h^*(q^*(t),v,t) \\ q^*(0) = q \end{cases} \qquad (3.36)$$

Since $h \in \mathscr{H}$, it is easily seen that for any $v \in \mathbb{R}^1$ the function $v^*(\cdot,v,\cdot)$ is continuous together with its partial derivative $\partial/\partial q\, v^*(\cdot,v,\cdot)$ in the whole (q,t) plane. From the general theory of ordinary differential equations (see, e.g., Ref. 12, Chaps. II and III) it follows that for all $(q,v) \in M$ there is a unique global solution, $q^* = q^*_{h,q,v}$ of Eq. (3.36). Denote by $u^*_{h,v,t}$ the map $q \in \mathbb{R}^1 \to q^*_{h,q,v}(t)$, $t \in \mathbb{R}^1$. Then for all $v,t \in \mathbb{R}^1$ $u^*_{h,v,t}$ is a diffeomorphism of $\mathbb{R}^1$ onto itself. ■

The uniqueness of the solution of the problem (3.1), (3.2) follows from the following lemma.

Lemma 3.IX. Let $g_0 \in \mathscr{G}$ and let $h \in \mathscr{H}$ be a solution of (3.1), (3.2). Then for any $v,t \in \mathbb{R}^1$

$$u^*_{h,v,t}(q) = u_{g_0,v,t}(q), \qquad q \in \mathbb{R}^1 \qquad (3.37)$$

$$h(q,v,t) = g_0\big(u^{*-1}_{h,v,t}(q),v\big)(\partial/\partial q)u^{*-1}_{h,v,t}(q), \qquad q \in \mathbb{R}^1 \qquad (3.38)$$

where $u_{g_0,v,t}$ is given by (3.19).

The proof of Lemma 3.IX is based on the following auxiliary statement.

Lemma 3.X. Let the conditions of Lemma 3.IX hold. Then

(i) for any $(q,v) \in M$ and $t \in \mathbb{R}^1$

$$h(u^*_{h,v,t}(q),v,t)(d/dq)u^*_{h,v,t}(q) = g_0(q,v) \qquad (3.39)$$

(ii) for any $(q,v) \in M$ and $t \in \mathbb{R}^1$

$$\mathbf{B}_{q^*(t)}h_t = S_{a_t}T_t^0\mathbf{B}_q g_0 \qquad (3.40)$$

where $q^*(t) = u^*(q,v,t)$, $a_t = q^*(t) - q - vt$.

We first show how Lemma 3.IX follows from Lemma 3.X. Let g_0 and h be as in Lemma 3.IX. Given $(q,v) \in M$ and $t \in \mathbb{R}^1$, we set $q^*(t) = u^*_{h,v,t}(q)$, $f^t = \mathbf{B}_{q^*(t)}h_t$ and $f^0 = \mathbf{B}_q g_0$. Using (3.19)–(3.21) and assertion (ii) of Lemma

3.X we find from (3.35) that

$$\begin{aligned} v_h^*(q^*(t), v, t) &= v + d\int_{\mathbb{R}^1} dv'\,(v - v')f^t(q^*(t), v') \\ &= v + d\int_{\mathbb{R}^1} dv'\,(v - v')T_t^0 f^0(q + vt, v') \\ &= v + d(\partial/\partial t)m_{g_0}(q, v, t) \end{aligned}$$

Integrating in t and using again (3.19), we get (3.37). The relation (3.37) together with assertion (i) of Lemma 3.X imply (3.38).

Now to prove Lemma 3.X we need the following result.

Lemma 3.XI. Let the conditions of Lemma 3.IX hold. Given (q_0, v_0), $(q, v) \in M$, set $q_0^*(t) = u^*_{h,v_0,t}(q_0)$, $q^*(t) = u^*_{h,v,t}(q)$. Then the following equality holds:

$$B_{h_t, q_0^*(t)}(q^*(t)) - B_{g_0, q_0}(q) - vt - \left[q_0^*(t) - q_0 - v_0 t\right] = 0 \qquad (3.41)$$

Proof of Lemma 3.XI. Integrating both sides of (3.1) in v we find that for all $q, t \in \mathbb{R}^1$

$$(\partial/\partial t)\sigma_{h_t}(q) + (\partial/\partial q)\zeta_{h_t}(q) = 0 \qquad (3.42)$$

which expresses the mechanical law of mass conservation.

Now notice that for $t = 0$ the left-hand side of (3.41) is 0. Hence, to prove (3.41) it is enough to check that the derivative of the left-hand side in t is 0. Using Definition 3.6 we find that this derivative is

$$\begin{aligned} &v^*(q^*(t), v, t)\left[1 - d\sigma_{h_t}(q^*(t))\right] - v^*(q_0^*(t), v_0, t)\left[1 - d\sigma_{h_t}(q_0^*(t))\right] \\ &\quad + d\int_{q_0^*(t)}^{q^*(t)} dq'\,(\partial/\partial t)\sigma_{h_t}(q') + v_0 - v \end{aligned} \qquad (3.43)$$

Taking into account (3.42) and the equality $v^*(q, v, t) = [v - d\zeta_{h_t}(q)][1 - d\sigma_{h_t}(q)]^{-1}$, it is readily seen that (3.43) is equal to 0. ■

Proof of Lemma 3.X. (i) Notice that Eq. (3.39) is obviously true for $t = 0$. Hence it is enough to prove that the derivative of the left-hand side in t is 0. Setting $q^*(t) = u^*_{h,v,t}(q)$ and using Eq. (3.1) we find for the derivative

$$\begin{aligned} &\left[(\partial/\partial t)h(\tilde{q}, v, t)|_{\tilde{q}=q^*(t)} + (\partial/\partial\tilde{q})h(\tilde{q}, v, t)|_{\tilde{q}=q^*(t)}\right](\partial/\partial q)q^*(t) \\ &\quad + h(q^*(t), v, t)\left[\partial/\partial\tilde{q}\, v^*(\tilde{q}, v, t)|_{\tilde{q}=q^*(t)}\right](\partial/\partial q)q^*(t) \\ &= \left(\left\{\partial/\partial t\, h(\tilde{q}, v, t) + \partial/\partial\tilde{q}\left[h(\tilde{q}, v, t)v^*(\tilde{q}, v, t)\right]\right\}|_{\tilde{q}=q^*(t)}\right)(\partial/\partial q)q^*(t) \\ &= 0 \end{aligned}$$

(ii) We set, as above $f^t = \mathbf{B}_{q^*(t)} h_t$, $f^0 = \mathbf{B}_q g_0$. We must prove that for any $(q,v) \in M$, $t \in \mathbb{R}^1$

$$f^t(q,v) = f^0(q - vt - a_t, v) \tag{3.44}$$

Since $u^*_{h,v,t}$ and $B_{h_t, q_0^*(t)}$ are both diffeomorphisms, their composition $B_{h_t, q_0^*(t)}(u^*_{h,v,t}(\cdot))$ is surjective. Thus, we can replace q in both sides of (3.44) by $B_{h_t, q_0^*(t)}(q^*(t))$. After this change of variables and using Lemma 3.XI we find that the right-hand side of (3.44) is equal to

$$f^0(B_{g_0, q_0}(q), v) = g_0(q,v)[1 - d\sigma_{g_0}(q)]^{-1}$$

On the other hand, according to Definitions 3.6, 3.7, the left-hand side of (3.41) is equal to

$$f^{(t)}(B_{h_t, q_0^*(t)}(q^*(t)), v) = h_t(q^*(t), v)[1 - d\sigma_{h_t}(q^*(t))]^{-1}$$

Taking into account the equality

$$(d/dq) u^*_{h,v,t}(q) = [1 - d\sigma_{g_0}(q)][1 - d\sigma_{h_t}(q^*(t))]^{-1}$$

which holds, because of Lemma 3.X, we see that (3.44) is equivalent to (3.39). This proves (ii). ■

4. THE CONVERGENCE THEOREM

Theorem 4.1. Let $g_0 \in \mathscr{G}$ (see Definition 3.5), and let $\{P^\epsilon, \epsilon > 0\}$ be a family of states such that

(i) $P^\epsilon(\mathscr{M}_d') = 1$ [where $\mathscr{M}_d'$ is given by Eq. (2.23)] for all $\epsilon > 0$;

(ii) for any $\epsilon > 0$ the 1-particle correlation measure $K^{(1)}_{P^\epsilon}$ is absolutely continuous with respect to the Lebesgue measure l on M and the 1-particle correlation function $k^{(1)}_{P^\epsilon}$ has the following properties: (a) for some $\varphi \in \mathscr{L}$ (see Definition 3.1),

$$k^{(1)}_{P^\epsilon}(q,v) < \varphi(v), \qquad (q,v) \in M \tag{4.1a}$$

and (b)

$$\lim_{\epsilon \to 0} k^{(1)}_{P^\epsilon}(\epsilon^{-1} q, v) = g_0(q,v), \qquad (q,v) \in M \tag{4.1b}$$

(iii) for any $t \in \mathbb{R}^1$ and any bounded interval J there are two nondecreasing functions, $r_{t,J}, s_{t,J} : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ such that $\lim_{u \to 0+} r_{t,J}(u) = \lim_{u \to 0+} s_{t,J}(u) = 0$, and for all $\epsilon > 0$, $t \in \mathbb{R}^1$ and $(q,v) \in \mathbb{R}^1 \times J$

$$P^\epsilon\Big[\Big\{|\epsilon n_X(\epsilon^{-1} q, v, \epsilon^{-1} t) - m_{g_0}(q,v,t)| > d^{-1} r_{t,J}(\epsilon)\Big\}\Big] < s_{t,J}(\epsilon) \tag{4.2}$$

where n_X is given by (2.26)–(2.27), (2.25) and m_{g_0} by (3.20)–(3.22).

Then, for any $t \in \mathbb{R}^1$, the measure G_t^ϵ on $(M, \mathscr{D})$ given by

$$G_t^\epsilon(C \times B) = \epsilon K^{(1)}_{P^\epsilon_{\epsilon^{-1}t}}(\epsilon^{-1}C \times B), \qquad C, B \subset \mathbb{R}^1 \tag{4.3}$$

converges in the vague topology as $\epsilon \to 0$ to the measure G_t on $(M, \mathscr{D})$ which is absolutely continuous with respect to l with a density $g_t = \mathscr{U}_t g_0$ given by Eq. (3.28) [which is by Theorem 3.VI the unique solution of the Cauchy problem (3.1), (3.2)].

Remark. It is easy to see that assumption (iii) of the theorem is equivalent to the following one, which is easier to verify:

(iii′) for any $\delta > 0$, $t \in \mathbb{R}^1$ and any bounded interval $J \subset \mathbb{R}^1$, the following relation holds uniformly in $(q, v) \in \mathbb{R}^1 \times J$

$$\lim_{\epsilon \to 0} P^\epsilon\left(\left\{|\epsilon n_X(\epsilon^{-1}q, v, \epsilon^{-1}t) - m_{g_0}(q, v, t)| > \right\}\right) = 0 \tag{4.2′}$$

Proof of Theorem 4.1. Let $t \in \mathbb{R}^1$ be fixed. For definiteness we assume $t > 0$ (for $t < 0$ the proof is similar). We should prove that for any set $E \subset M$ such that $E = I \times J$, where I and J are bounded intervals,

$$\lim_{\epsilon \to 0} G_t^\epsilon(E) = \int_J dv \int_{u^{-1}_{g_0,v,t}(I)} dq\, g_0(q, v) \tag{4.4}$$

where the map $u_{g_0,v,t} : \mathbb{R}^1 \to \mathbb{R}^1$ is defined by Eqs. (3.19)–(3.22).

The proof of Eq. (4.4) is based on a lemma which we give below. Let the intervals $I = [q', q'']$ and $J = [v', v'']$ be fixed. From now on we shall write r and s instead of $r_{t,J}$ and $s_{t,J}$, since t and J are fixed. Choose two nondecreasing functions $\beta, \delta : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ such that $\lim_{u \to 0+} \beta(u) = \lim_{u \to 0+} \delta(u) = 0$, and, moreover, $\delta(u) > r(u)$, $u > 0$, and $\lim_{u \to 0+} (\beta(u))^{-1} s(u) = 0$ (it is clear that such functions exist). Set furthermore $l(\epsilon) = [(\beta(\epsilon))^{-1}(v'' - v')]$, where [ ] denotes the integer part, $\check{l}(\epsilon) = 2(l(\epsilon) + 1)$ $s(\epsilon)$, and $v_i = v' + i\beta(\epsilon)$, $J_i = [v_i, v_{i+1}]$, $i = 0, 1, \ldots, l(\epsilon) + 1$. Furthermore we set

$$q_1^{\wedge\epsilon}(i) = \epsilon^{-1} u^{-1}_{g_0, v_{i+1}, t}(q' - \delta(\epsilon)) \tag{4.5a}$$

$$q_2^{\wedge\epsilon}(i) = \epsilon^{-1} u^{-1}_{g_0, v_i, t}(q'' + \delta(\epsilon)) \tag{4.5b}$$

$$q_1^{\vee\epsilon}(i) = \epsilon^{-1} u^{-1}_{g_0, v_i, t}(q' + \delta(\epsilon)) \tag{4.6a}$$

$$q_2^{\vee\epsilon}(i) = \epsilon^{-1} u^{-1}_{g_0, v_{i+1}, t}(q'' - \delta(\epsilon)) \tag{4.6b}$$

Owing to the monotonicity property of the maps $u_{g,v,t}$, $v \in \mathbb{R}^1$ (see Corollary 3.V), we obtain that for any $\epsilon > 0$, $q_1^{\wedge\epsilon}(i) < q_2^{\wedge\epsilon}(i)$ and, if ϵ is small enough, $q_1^{\vee\epsilon}(i) < q_2^{\vee\epsilon}(i)$, $i = 0, 1, \ldots, l(\epsilon) + 1$. Consider the intervals

$$I_\epsilon^{\wedge(i)} = \left[q_1^{\wedge\epsilon}(i), q_2^{\wedge\epsilon}(i)\right], \qquad I_\epsilon^{\vee(i)} = \left[q_1^{\vee\epsilon}(i), q_2^{\vee\epsilon}(i)\right] \tag{4.7}$$

and the sets

$$E_t^{\wedge\epsilon} = \bigcup_{i=0}^{l(\epsilon)} \left(I_\epsilon^{\wedge(i)} \times J_i\right), \qquad E_t^{\vee\epsilon} = \bigcup_{i=0}^{l(\epsilon)} \left(I_\epsilon^{\vee(i)} \times J_i\right) \tag{4.8}$$

Lemma 4.2. Under the assumptions of Theorem 4.1 there exists a $\mathfrak{M}$-measurable set $\mathscr{M}_d^\epsilon \subset \mathscr{M}_d'$ such that for any $\epsilon > 0$ (i) $P^\epsilon(\mathscr{M}_d' \setminus \mathscr{M}_d^\epsilon) < \bar{l}(\epsilon)$, (ii) for any $X \in \mathscr{M}_d^\epsilon$ the following inequalities hold

$$|X \cap E_t^{\vee\epsilon}| < |T^d_{\epsilon^{-1}t} X \cap E_\epsilon| < |X \cap E_t^{\wedge\epsilon}|$$

where $E_\epsilon = I_\epsilon \times J$, $I_\epsilon = \epsilon^{-1} I$.

Proof. Using the definition of the maps $w_{X,v,\tau}$ [see (2.24)–(2.27)] and $u_{g_0,v,t}$ [see (3.19)–(3.22)] and assumption (iii) of Theorem 4.1, we conclude that for any $\epsilon > 0$ and $i = 0, 1, \ldots, l(\epsilon) + 1$ there exists an $\mathfrak{M}$-measurable set $\mathscr{M}_{d,i}^\epsilon(+) \subset \mathscr{M}_d'$ such that $P^\epsilon(\mathscr{M}_d' \setminus \mathscr{M}_{d,i}^\epsilon(+)) < s(\epsilon)$, and for all $X \in \mathscr{M}_{d,i}^\epsilon(+)$ the following inequalities hold:

$$\left|w_{X,v_{i+1},\epsilon^{-1}t}\left(q_1^{\wedge\epsilon}(i)\right) - \epsilon^{-1} u_{g_0,v_{i+1},t}\left(\epsilon q_1^{\wedge\epsilon}(i)\right)\right| < \epsilon^{-1} r(\epsilon)$$

$$\left|w_{X,v_i,\epsilon^{-1}t}\left(q_2^{\wedge\epsilon}(i)\right) - \epsilon^{-1} u_{g_0,v_i,t}\left(\epsilon q_2^{\wedge\epsilon}(i)\right)\right| < \epsilon^{-1} r(\epsilon)$$

According to the definitions of $q_1^{\wedge\epsilon}(i)$ and $q_2^{\wedge\epsilon}(i)$ [see Eq. (4.5a, b)],

$$\epsilon^{-1} u_{g_0,v_{i+1},t}\left(\epsilon q_1^{\wedge\epsilon}(i)\right) = \epsilon^{-1} q' - \epsilon^{-1} \delta(\epsilon)$$

$$\epsilon^{-1} u_{g_0,v_i,t}\left(\epsilon q_2^{\wedge\epsilon}(i)\right) = \epsilon^{-1} q'' + \epsilon^{-1} \delta(\epsilon)$$

Hence, for any $\epsilon > 0$ and $i = 0, 1, \ldots, l(\epsilon) + 1$, for all $X \in \mathscr{M}_{d,i}^\epsilon(+)$

$$w_{X,v_{i+1},\epsilon^{-1}t}\left(q_1^{\wedge\epsilon}(i)\right) < \epsilon^{-1} q' - \epsilon^{-1}\delta(\epsilon) + \epsilon^{-1} r(\epsilon) < \epsilon^{-1} q'$$

$$w_{X,v_i,\epsilon^{-1}t}\left(q_2^{\wedge\epsilon}(i)\right) > \epsilon^{-1} q'' + \epsilon^{-1}\delta(\epsilon) - \epsilon^{-1} r(\epsilon) > \epsilon^{-1} q''$$

Owing to the monotonicity property of the maps $w_{X,v,\tau}$ (see the end of Section 2), the two last inequalities imply that for any $\epsilon > 0$ and $i = 0, 1, \ldots, l(\epsilon) + 1$ for all $X \in \mathscr{M}_{d,i}^\epsilon(+)$

$$|T^d_{\epsilon^{-1}t} X \cap (I_\epsilon \times J_i)| \leqslant |X \cap \left(I_\epsilon^{\wedge(i)} \times J_i\right)| \tag{4.9}$$

Let $\mathscr{M}_d^\epsilon(+)$ be the intersection $\bigcap_{i=0}^{l(\epsilon)+1} \mathscr{M}_{d,i}^\epsilon(+)$. Then

$$P^\epsilon\left(\mathscr{M}_d' \setminus \mathscr{M}_d^\epsilon(+)\right) < (1/2)\bar{l}(\epsilon)$$

and for any $X \in \mathscr{M}_d^\epsilon(+)$ and all $i = 0, \ldots, l(\epsilon) + 1$ the inequality (4.9) holds. Hence, for any $\epsilon > 0$ and $X \in \mathscr{M}_d^\epsilon(+)$

$$|T^d_{\epsilon^{-1}t} X \cap E_\epsilon| \leqslant |X \cap E_t^{\wedge\epsilon}|$$

Proceeding in the same way one can see that for any $\epsilon > 0$ there is a set $\mathscr{M}_d^\epsilon(-) \subset \mathscr{M}_d'$ such that $P^\epsilon(\mathscr{M}_d' \backslash \mathscr{M}_d^\epsilon(-)) < (1/2)\bar{l}(\epsilon)$ and for any $X \in \mathscr{M}_d^\epsilon(-)$

$$|T^d_{\epsilon^{-1}t} X \cap E_\epsilon| \geqslant |X \cap E_t^{\vee\epsilon}|$$

Taking $\mathscr{M}_d^\epsilon = \mathscr{M}_d^\epsilon(-) \cap \mathscr{M}_d^\epsilon(+)$ we obtain the assertion of Lemma 4.II. Now pass to the proof of (4.4). First we show that

$$\limsup_{\epsilon\to 0} G_t^\epsilon(E) \leqslant \int_J dv \int_{u^{-1}_{g_0,v,t}(I)} dq\; g_0(q,v) \tag{4.10}$$

Observe that in force of Lemma 4.II

$$\begin{aligned} K^{(1)}_{P^\epsilon_{\epsilon^{-1}t}}(E_\epsilon) &= \int_{\mathscr{M}_d^\epsilon} P^\epsilon(dX)\,|T^d_{\epsilon^{-1}t} X \cap E_\epsilon| + \int_{\mathscr{M}_d' \backslash \mathscr{M}_d^\epsilon} P^\epsilon(dX)\,|T^d_{\epsilon^{-1}t} X \cap E_\epsilon| \\ &\leqslant \int_{\mathscr{M}_d^\epsilon} P^\epsilon(dX)\,|X \cap E_t^{\wedge\epsilon}| + \epsilon^{-1} d(q''-q')\,P^\epsilon(\mathscr{M}_d' \backslash \mathscr{M}_d^\epsilon) \\ &\leqslant K^{(1)}_{P^\epsilon}(E_t^{\wedge\epsilon}) + \epsilon^{-1} d|q''-q'|\,\bar{l}(\epsilon) \end{aligned}$$

whence

$$G_t^\epsilon(E) \leqslant \epsilon K^{(1)}_{P^\epsilon}(E_t^{\wedge\epsilon}) + d^{-1}|q''-q'|\,\bar{l}(\epsilon) \tag{4.11}$$

Given $v \in J$, we set $\bar{v}_\epsilon = ([\beta(\epsilon)^{-1}(v-v')]+1)\beta(\epsilon)$ and

$$a_\epsilon(v) = u^{-1}_{g_0,\bar{v}_\epsilon,t}(q' - \delta(\epsilon)), \qquad b_\epsilon(v) = u^{-1}_{g_0,\bar{v}_\epsilon,t}(q'' + \delta(\epsilon))$$

Then

$$\epsilon K^{(1)}_{P^\epsilon}(E_t^{\wedge\epsilon}) = \sum_{i=0}^{l(\epsilon)} \int_{\epsilon I^{\wedge(i)} \times J_i} dq\,dv\,k^{(1)}_{P^\epsilon}(\epsilon^{-1}q, v) = \int_J dv \int_{a_\epsilon(v)}^{b_\epsilon(v)} dq\,k^{(1)}_{P^\epsilon}(\epsilon^{-1}q, v)$$

Because of the continuity property of the maps $u_{g_0,v,t}$, for any $v \in \mathbb{R}^1$

$$\lim_{\epsilon\to 0} a_\epsilon(v) = u^{-1}_{g_0,v,t}(q'), \qquad \lim_{\epsilon\to 0} b_\epsilon(v) = u^{-1}_{g_0,v,t}(q'')$$

From condition (ii) of Theorem 4.I and the dominated convergence theorem it follows that

$$\lim_{\epsilon\to 0} \epsilon K^{(1)}_{P^\epsilon}(E_t^{\wedge\epsilon}) = \int_J dv \int_{u^{-1}_{g_0,v,t}(q')}^{u^{-1}_{g_0,v,t}(q'')} dq\; g_0(q,v) \tag{4.12}$$

Together with (4.11) this gives (4.10) since $\lim_{\epsilon\to 0} \bar{l}(\epsilon) = 0$ and $u^{-1}_{g_0,v,t}(I) = [u^{-1}_{g_0,v,t}(q'), u^{-1}_{g_0,v,t}(q'')]$.

The inverse inequality

$$\liminf_{\epsilon\to 0} G_t^\epsilon(E) \geqslant \int_J dv \int_{u^{-1}_{g_0,v,t}(I)} dq\; g_0(q,v) \tag{4.13}$$

may be proved in a similar way with replacing $E_t^{\wedge\epsilon}$ by $E_t^{\vee\epsilon}$. The inequalities (4.10) and (4.13) imply (4.4). Theorem 4.I is proved. ■

5. FAMILIES OF STATES SATISFYING THE CONDITIONS OF THE CONVERGENCE THEOREM

In this section we give examples of families of states for which the conditions of Theorem 4.I hold. The states will be Gibbs states with a potential Φ_ϵ such that $\Phi_\epsilon^{(n)} = 0$ for $n \geqslant 3$, and $\Phi_\epsilon^{(2)}$ does not depend on the particle velocities (see Definitions 2.3, 2.4). These restrictions are not essential and are used to simplify the technical details.

Let $g_0 \in \mathscr{G}_\psi$ for some function $\psi \in \mathscr{L}$ (see Definition 3.5). We set

$$V(q,v) = -\ln g_0(q,v) \qquad (q,v) \in M \tag{5.1}$$

Furthermore let a function $U: \mathbb{R}^1 \times \mathbb{R}^1_+ \to \overline{\mathbb{R}}^1$ be given, with the following properties.

Condition 5.1. (i) $U(q,r) = +\infty$ if $0 \leqslant r \leqslant d$, and $U(q,r) < +\infty$ if $r > d$; (ii) there is a constant $B > -\infty$ such that $U(q,r) \geqslant B$, $(q,r) \in \mathbb{R}^1 \times \mathbb{R}^1_+$; (iii) there are constants $c_0 > 0$, $c_1 > d$ and $\kappa_0 > 0$ such that $|U(q,r)| < c_0 r^{-(3+\kappa_0)}$ for $r > c_1$; (iv) for any $r > d$ the function $U(\cdot,r): \mathbb{R}^1_+ \to \overline{\mathbb{R}}^1$ is of class C^1 and $|(\partial/\partial q)U(q,r)| < c_2 r^{-(2+\kappa_0)}$ for some $c_2 > 0$.

Remark. The reader may have in mind the simple case in which the function $U(q,r)$ does not depend on $q \in \mathbb{R}^1$, for which the proofs are simpler.

For a given function $\mu: \mathbb{R}^1 \to \mathbb{R}^1$ consider the family of auxiliary translation invariant c potentials $\{\overline{\Psi}_{(q)}, q \in \mathbb{R}^1\}$ which are given by

$$\begin{aligned} \overline{\Psi}^{(1)}_{(q)}(q_1) &= \text{const} = \mu(q) - \ln \sigma_{g_0}(q), && q_1 \in \mathbb{R}^1 \\ \overline{\Psi}^{(2)}_{(q)}(\{(q_1, q_2)\}) &= U(q, |q_1 - q_2|), && q_1, q_2 \in \mathbb{R}^1, q_1 \neq q_2 \\ \overline{\Psi}^{(n)}_{(q)} &\equiv 0, && n \geqslant 3 \end{aligned} \tag{5.2}$$

It is well known (see Refs. 13, 14, and 15) that, given an arbitrary function μ, for any $q \in \mathbb{R}^1$ there is a unique Gibbs c state $Q^{(q)} = Q^{(\overline{\Psi}_{(q)})}$ with c potential $\overline{\Psi}_{(q)}$, and $Q^{(q)}$ is a translation-invariant c state. We are interested in a particular choice of the function μ which is indicated in the following proposition.

Proposition 5.I. For any $q \in \mathbb{R}^1$ there is a unique value $\mu(q)$ such that the following equality holds:

$$\exp[-\mu(q)]\hat{k}^{(1)}_{Q^{(q)}} = 1 \tag{5.3}$$

[see Eq. (2.13)]. Furthermore the function $\mu(q)$, $q \in \mathbb{R}^1$, is continuous and bounded from below.

Proof. Existence and uniqueness of $\mu(q)$ follow from Ref. 16. In fact, for any $q \in \mathbb{R}^1$ with $\sigma_{g_0}(q) > 0$ the map

$$\mu(q) \in \mathbb{R}^1 \to \exp[-\mu(q)]\hat{k}^{(1)}_{Q^{(q)}} \in \left[0, (\sigma_{g_0}(q)d)^{-1}\right]$$

defines a nonincreasing function of the variable $\mu(q)$, as it follows immediately from the definition of Gibbs c state (Definition 2.4). According to Ref. 16 this function is analytic, and hence strictly increasing with $\lim_{\mu(q)\to+\infty} \exp[-\mu(q)]\hat{k}^{(1)}_{Q^{(q)}} = 0$, $\lim_{\mu(q)\to-\infty} \exp[-\mu(q)]\hat{k}^{(1)}_{Q^{(q)}} = [\sigma_{g_0}(q)d]^{-1} > 1$. Therefore there is a unique $\mu(q)$ such that Eq. (5.3) holds. If $\sigma_{g_0}(q) = 0$ we set $\mu(q) = 0$, taking into account that $\hat{k}^{(1)}_{Q^{(q)}} = 1$ in this case.

Furthermore, it follows from the analyticity result of Ref. 16 and condition 5.1 (iv) that $\mu(q)$ is continuous, and conditions 5.1 (i)–(iii) and Eq. (2.13a) give the inequality

$$\hat{k}^{(1)}_{Q^{(q)}} < 1 + \exp\left\{-B\left[(c_1 - d)d^{-1} + 1\right] + (c_0/d)\int_{c_1}^{\infty} dr\, r^{-(3+\kappa_0)}\right\} \tag{5.4}$$

Henceforth the symbol $\mu(q)$ will denote the value mentioned in Proposition 5.1.

An important property of the c states $Q^{(q)}$, $q \in \mathbb{R}^1$, is that they are uniformly approximated by the conditional Gibbs states for finite volumes (see Definition 2.4). In particular, setting

$$G^{(q,s,q')}(\cdot \mid Y) = G^{(\Psi_{(q)}, I^{(s)}+q')}(\cdot \mid Y), \qquad Y \in \mathcal{N}_d\left(\left(I^{(s)} + q'\right)^c\right) \tag{5.5}$$

where $I^{(s)} = [-s, s]$, one has

$$\lim_{s\to\infty} \sup_{q\in\mathbb{R}^1} \sup_{Y\in\mathcal{N}_d((I^{(s)}+q')^c)} |\hat{k}^{(1)}_{Q^{(q)}} - \hat{k}^{(1)}_{G^{(q,s,q')}(\cdot \mid Y)}(q')| = 0 \tag{5.6}$$

(note that the internal sup does not depend on q'). The proof of Eq. (5.6) follows immediately from the arguments of Refs. 13, 14, and 15.

Consider now the potentials Φ_ϵ, $\epsilon > 0$, given by

$$\begin{aligned} \Phi^{(1)}_\epsilon(q_1, v_1) &= \mu(\epsilon q_1) + V(\epsilon q_1, v_1), \qquad (q_1, v_1) \in M \\ \Phi^{(2)}_\epsilon(\{(q_1, v_1), (q_2, v_2)\}) &= \tfrac{1}{2}\left[U(\epsilon q_1, |q_1 - q_2|) + U(\epsilon q_2, |q_1 - q_2|)\right], \\ &\qquad (q_i, v_i) \in M, \quad i = 1,2, \quad (q_1, v_1) \neq (q_2, v_2) \\ \Phi^{(n)}_\epsilon &\equiv 0, \qquad n \geqslant 3 \end{aligned} \tag{5.7}$$

We can now state our convergence theorem.

Theorem 5.II. Let the potential Φ_ϵ, $\epsilon > 0$, be given by Eq. (5.7), where the function V is given by Eq. (5.1), the function U satisfies assumption 5.1, and μ is as stated in Proposition 5.I. Then for any $\epsilon > 0$ there is a unique Gibbs state with potential Φ_ϵ, and the family of states $\{P^\epsilon, \epsilon > 0\}$ satisfies the assumptions of Theorem 4.I.

Proof. Existence and uniqueness of the Gibbs state with potential Φ_ϵ may be proved, using condition 5.1, along the lines of the papers of Refs. 13, 14, and 15. Since the necessary modifications involve only a few technical details, we shall omit the proof.

In order to prove that the assumptions of Theorem 4.I are satisfied, we state some properties of the states P^ϵ which are needed. Such properties are essentially related to the construction of the states P^ϵ, as it is done in Refs. 13, 14, and 15, and may be proved following the same pattern, with the necessary technical modifications.

Property (A). The states P^ϵ, $\epsilon > 0$, are concentrated on the set $\overline{\mathscr{M}}_d$ [Eq. (2.17)], i.e., $P^\epsilon(\overline{\mathscr{M}}_d) = 1$. This property follows immediately from condition 5.1 (i) and the fact that Gibbs states are locally absolutely continuous.

Property (B). For any $\epsilon > 0$ the 1-particle correlation function has the form

$$k_{P^\epsilon}^{(1)}(q,v) = g_0(\epsilon q, v)\exp[-\mu(\epsilon q)]\hat{k}_P^{(1)}(q) \tag{5.8}$$

where $\hat{k}_P$ is defined by Eq. (2.13), and is uniformly bounded:

$$\hat{k}_{P^\epsilon}^{(1)}(q) < \text{RHS of Eq. (5.4)} \tag{5.9}$$

This property follows easily from Eq. (5.7) and conditions 5.1 (i)–(iii).

Property (C). The particle density α_{P^ϵ} [see Eq. (2.7)–(2.8′)] is bounded away from d^{-1}:

$$\sup_{\epsilon > 0, q \in \mathbb{R}^1} \alpha_{P^\epsilon}(q) = \bar{\alpha} < d^{-1} \tag{5.10}$$

This follows from the condition $g_0 \in \mathscr{G}$, from the boundedness of μ from below (see Proposition 5.I), and conditions 5.1 (i)–(iii), which hold uniformly in $q \in \mathbb{R}^1$.

Property (D). Setting

$$G^{(\epsilon,s,q)}(\cdot\,|\,X) = G^{(\Phi_\epsilon, I^{(s)}+q)}(\cdot\,|\,X), \qquad X \in \mathscr{M}_d^+\left((I^{(s)}+q)^c\right) \tag{5.11}$$

where $I^{(s)} = [-s, s]$, one has, in analogy with Eq. (5.6),

$$\lim_{s\to\infty} \sup_{\epsilon>0, q\in\mathbb{R}^1} \sup_{X\in\mathscr{M}_d^+((I^{(s)}+q)^c)} |\hat{k}_P^{(1)}(q) - \hat{k}_{G^{(\epsilon,s,q)}(\cdot\,|\,X)}^{(1)}(q)| = 0 \tag{5.12}$$

Property (E). There is a constant $c \in \mathbb{R}_+^1$ such that the random variable $\xi_{I\times J}$ [defined by Eq. (2.4)] for any bounded interval I and for all subsets $J \subset \mathbb{R}^1$ satisfies the inequality

$$\sup_{\epsilon>0} \mathbb{D}_{P^\epsilon}\xi_{I\times J} < c|I| \tag{5.13}$$

This property follows in a standard way from the fact that the uniform mixing coefficient $\delta(s)$, $s \in \mathbb{R}^1_+$, defined by

$$\delta(s) = \sup_{\substack{\epsilon>0, s_0>0 \\ q\in\mathbb{R}^1}} \operatorname{sup\,ess}_{X\in\mathscr{M}}(P^\epsilon) \sup_{A\in\mathfrak{M}(I^{(s_0)}+q)} |P^\epsilon(A) - P^\epsilon_{\mathfrak{M}((I^{(s+s_0)}+q)^c)}(A \mid X)|$$

is integrable, i.e., $\int_{\mathbb{R}^1_+} \delta(s)\,ds < \infty$.

We pass now to verifying the assumptions of Theorem 4.I.

We start with assumption (i). From properties (A), (C), and (E) above, using the Chebyshev inequality, it follows easily that $P^\epsilon(\mathscr{M}_d^+) = 1$. Therefore, to check assumption (i) we need to prove that for P^ϵ-a.a. $X \in \mathscr{M}$ and all $\tau \in \mathbb{R}^1$, $(q,v) \in M$, the quantities $n_X^\pm(q,v,t)$ given by Eq. (2.27) are finite. A simple analysis shows that they are finite for all $\tau \in \mathbb{R}^1$ and $(q,v) \in M$, if they are finite when τ and (q,v) belong to some countable sets which are dense in $\mathbb{R}^1, M$, respectively [say, for rational τ and (q,v)]. The latter fact will be proved if we prove that for any $\epsilon > 0$, $\tau \in \mathbb{R}^1$, and $(q,v) \in M$, the quantities $n_X^\pm(q,v,\tau)$ are finite for P^ϵ-a.a. X. To prove this we shall find two sequences of (proper) random variables $\{\eta_j^\pm = \eta_{q,v,j}^\pm$, $j = 1,2,\ldots\}$ such that $\lim_{j\to\infty} P^\epsilon(\{n_X^\pm(q,v,\tau) < \eta_j^\pm\}) = 1$ for any $\epsilon > 0$.

For definiteness we assume $\tau > 0$ and consider the case "+." The reader can extend the arguments to the other cases without difficulty. We set

$$\eta_j^+(X) = \sum_{j'=1}^\infty \xi_{I_j(j')\times J(j')}(X), \qquad X \in \mathscr{M}_d^+ \tag{5.14}$$

where

$$I_j(j') = [q, q + s_j(j')], \qquad J(j') = [v - 2^{j'} + 1, v - 2^{j'-1} + 1)$$

and

$$s_j(j') = \max\left[\frac{\tau 2^{j'}}{1 - \bar{\alpha}d}, j\right], \qquad j, j' = 1, 2, \ldots$$

$\bar{\alpha}$ being defined by Eq. (5.10). For any $j = 1,2,\ldots$ and $\epsilon > 0$, using property (B), the uniform boundedness of μ from below and the fact that $g_0 \in \mathscr{G}$ we have

$$\begin{aligned}\mathbb{E}_{P^\epsilon}\eta_j^+ &= \sum_{j'=1}^\infty \mathbb{E}_{P^\epsilon}\xi_{I_j(j')\times J(j')} \\ &\leqslant \sum_{j'=1}^\infty \int_{I_j(j')\times J(j')} dq'\,dv'\,k_{P^\epsilon}^{(1)}(q',v') \\ &\leqslant \text{const} + \text{const}'\tau(1 - \bar{\alpha}d)^{-1} \sum_{j'=1}^\infty 2^{j'} \int_{J(j')} dv\,\psi(v) < \infty \end{aligned} \tag{5.15}$$

(Here the constants may depend on j). Hence $P^{\epsilon}(\{\eta_j^+ < \infty) = 1$. To complete the proof we introduce the representation

$$n_X^+(q,v,\tau) = \sum_{j'=1}^{\infty} n_{X,J(j')}^+(q,v,\tau) \tag{5.16}$$

where we introduced the notation

$$n_{X,J}^+(q,v,\tau) = |(X_{(q,v)}^R(\tau))_{[q,+\infty)\times J}|, \qquad J \subset (-\infty, v] \tag{5.17}$$

We need at this point some simple properties of $n_{X,J}^+(q,v,\tau)$ for a bounded $J = [a_1, a_2] \subset (-\infty, v]$. It is easy to see that $n_{X,J}^+(q,v,\tau)$ is equal to the number of values of $i \in \mathbb{Z}_+^1$ for which (a) $v_i^{(q)}(X) \in J$ and (b) $(q_i^{(q)}(X) - id + \tau v_i^{(q)}(X), v_i^{(q)}(X)) < ((q + \tau v), v)$. This is less or equal to the number of values of $i \in \mathbb{Z}_+^1$ for which condition (a) holds and (b′) $q_i^{(q)}(X) - id < q + \tau v - \tau a_1$, which in its turn is bounded from above by the quantity

$$\inf\{|X_{[q,q+s]\times J}| : s \geqslant 0,\, s - d|X_{[q,q+s]\times \mathbb{R}^1}| > \tau(v - a_1) - d\}$$

Now, according to the Chebyshev inequality and property (E) [see Eq. (5.13)], for any $\epsilon, \gamma > 0$ we have

$$P^{\epsilon}\left(\left\{X \in \mathscr{M}_d^+ : \left||X_{[q,q+s]\times\mathbb{R}^1}| - \int_q^{q+s} dq' \int_{\mathbb{R}^1} dv'\, k_{P^\epsilon}^{(1)}(q',v')\right| > \gamma\right\}\right) < \frac{cs}{\gamma^2} \tag{5.18}$$

Therefore, if for some s and γ

$$s - d\int_q^{q+s} dq' \int_{\mathbb{R}^1} dv'\, k_{P^\epsilon}^{(1)}(q',v') - \gamma d > \tau(v - a_1) - d \tag{5.19}$$

then

$$P^{\epsilon}(\{X \in \mathscr{M}_d^+ : n_{X,J}^+(q,v,\tau) > |X_{q,q+s\times J}|\}) < \frac{cs}{\gamma^2} \tag{5.20}$$

In a similar way one can prove that, if for some s and γ

$$s - d\int_q^{q+s} dq' \int_{\mathbb{R}^1} dv'\, k_P^{(1)}(q',v') + \gamma d < \tau(v - a_2) - d \tag{5.21}$$

then

$$P^{\epsilon}(\{X \in \mathscr{M}_d^+ : n_{X,J}^+(q,v,\tau) < |X_{q,q+s\times J}|\}) < \frac{cs}{\gamma^2} \tag{5.22}$$

Take now $s = s_j(j') + (s_j(j'))^{7/8}$, $\gamma = (s_j(j'))^{2/3}$. It is easy to check that for j large enough the bound (5.19) holds when J is replaced by $J(j')$ for all

$j' = 1, 2, \ldots$. Therefore

$$P^\epsilon(\{X \in \mathscr{M}_d^+ : n_X^+(q,v,r) > \eta_j^+\})$$
$$\leqslant \sum_{j'=1}^{\infty} P^\epsilon(\{X \in \mathscr{M}_d^+ : n_{X,J(j')}^+(q,v,\tau) > |X_{I_j(j') \times J(j')}|\})$$
$$\leqslant c \sum_{j'=1}^{\infty} s_j(j')^{-1/3} \tag{5.23}$$

The right-hand side of Eq. (5.23) tends to 0 when $j \to \infty$. This proves assumption (i).

To check assumption (ii) of Theorem 4.I, observe that inequality (4.1a) follows from property (B), the uniform boundedness of μ from below, and the fact that $g_0 \in \mathscr{G}_\psi$ (ϕ can be taken proportional to ψ). As for Eq. (4.1b), in view of Eq. (5.3), it is enough to prove that for any $q \in \mathbb{R}^1$

$$\lim_{\epsilon \to 0} \hat{k}_{P^\epsilon}^{(1)}(\epsilon^{-1}q) = \hat{k}_{Q^{(q)}}^{(1)} \tag{5.24}$$

Using Eq. (5.6) and property (D), we reduce the problem to proving that for any $q \in \mathbb{R}^1$ and $s > 0$

$$\lim_{\epsilon \to 0} \sup_{X \in \mathscr{M}_d^+((I^{(s)} + \epsilon^{-1}q)^c)} |\hat{k}^{(1)}_{G^{(\epsilon,s,\epsilon^{-1}q)}(\cdot\,|X)}(\epsilon^{-1}q) - \hat{k}^{(1)}_{G^{(q,s,q)}(\cdot\,|S_{(\epsilon^{-1}-1)q}Y_X)}(q)| = 0 \tag{5.25}$$

where $Y_X = \{q \in \mathbb{R}^1 : (q,v) \in X$ for some $v \in \mathbb{R}^1\}$. Equality (5.25) is verified in a straightforward way [one uses here condition 5.1 (iv) and the continuity properties of the functions μ (see Proposition 5.I) and g_0].

It remains to check assumption (iii) of Theorem 4.I, or, equivalently, assumption (iii′) [see (4.2′)]. We have to prove that for any $\gamma > 0$, $t \in \mathbb{R}^1$ and bounded interval J the following relations hold uniformly in $(q,v) \in \mathbb{R}^1 \times J$:

$$\lim_{\epsilon \to 0} P^\epsilon\Big(\Big\{|\epsilon n_X^\pm(\epsilon^{-1}q, v, \epsilon^{-1}t) - m_{g_0}^\pm(q,v,t)| > \gamma\Big\}\Big) = 0 \tag{5.26}$$

[see (3.21)]. As above, we consider the case "+" and $t > 0$. The reader can extend the proof to the other cases in a straightforward way. Given $(q,v) \in \mathbb{R}^1 \times J$, we set

$$b_p^\epsilon = p[\ln(1/\epsilon)]^{-1}, \qquad \epsilon \in (0,1), \quad p = 0, 1, \ldots, \rho(\epsilon) = \Big\{[\ln(1/\epsilon)]^2\Big\} \tag{5.27}$$

$$\Delta_p^\epsilon = (v - b_{p+1}^\epsilon, v - b_p^\epsilon], \qquad p = 0, 1, \ldots, \rho(\epsilon); \quad \bar{\Delta}^\epsilon = (-\infty, v - b_{\rho(\epsilon)+1}^\epsilon] \tag{5.27′}$$

Clearly $n_X^+(\epsilon^{-1}q, v, \epsilon^{-1}t) = \sum_{p=0}^{\rho(\epsilon)} n_{X,\Delta_p^\epsilon}^+(\epsilon^{-1}q, v, \epsilon^{-1}t) + n_{X,\bar{\Delta}^\epsilon}^+(\epsilon^{-1}q, v, \epsilon^{-1}t)$. We check, first of all, that for any $\gamma > 0$

$$\lim_{\epsilon\to 0} P^\epsilon\Big(\big\{\epsilon n_{X,\bar{\Delta}^\epsilon}^+(\epsilon^{-1}q, v, \epsilon^{-1}t) > \gamma\big\}\Big) = 0 \tag{5.28}$$

uniformly in $(q,v) \in \mathbb{R}^1 \times J$. To prove this we introduce, in analogy with Eq. (5.14), the random variables $\eta_{j,\epsilon}^+$ replacing q by $\epsilon^{-1}q$, τ by $\epsilon^{-1}t$, and v by $v - b_{\rho(\epsilon)}^\epsilon$. Taking j large enough, we can make the probability

$$P^\epsilon\Big(\big\{X \in \mathscr{M}_d' : n_{X,\bar{\Delta}^\epsilon}^+(\epsilon^{-1}q, v, \epsilon^{-1}t) > \eta_{j,\epsilon}^+(X)\big\}\Big)$$

arbitrarily small [see Eq. (5.23)]. Furthermore, since $b_{\rho(\epsilon)}^\epsilon \to \infty$, i.e., $v - b_{\rho(\epsilon)}^\epsilon \to -\infty$, the expectation $\epsilon \mathbb{E}_{P^\epsilon} \eta_{j,\epsilon}^+$ vanishes as $\epsilon \to 0$ [see inequality (5.15)]. Hence, because of the Chebyshev inequality, $P^\epsilon(\{\epsilon\eta_{j,\epsilon}^+ > \gamma\})$ vanishes as $\epsilon \to 0$ for any $\gamma > 0$. From what we said above it follows that the limit in Eq. (5.28) is arbitrarily small, and hence is zero. Note that all estimates and limit relations are uniform in $(q,v) \in \mathbb{R}^1 \times J$.

Therefore it suffices to prove that

$$\lim_{\epsilon\to 0} P^\epsilon\left(\left\{\left|\epsilon \sum_{j=0}^{P(\epsilon)} n_{X,\Delta_j^\epsilon}^+(\epsilon^{-1}q, v, \epsilon^{-1}t) - m_{g_0}^+(q,v,t)\right| > \gamma\right\}\right) = 0 \tag{5.29}$$

For any $q \in \mathbb{R}^1$ we define the function $W_\epsilon = W_{\epsilon,q} : \mathbb{R}_+^1 \to [\epsilon^{-1}q, \infty)$ as the solution of the equation

$$\int_{\epsilon^{-1}q}^{W_\epsilon(u)} dq' \left[1 - \alpha_{P^\epsilon}(q')d\right] = u, \qquad u \in \mathbb{R}_+^1 \tag{5.30}$$

In view of property (C) above, the solution $W_\epsilon(u)$ of Eq. (5.30) exists and is unique for all $u \in \mathbb{R}_+^1$. Furthermore the following inequalities hold:

$$u(1 - \bar{\alpha}d)^{-1} \geqslant W_\epsilon(u) - \epsilon^{-1}q > u, \qquad u \in \mathbb{R}_+^1 \tag{5.31}$$

We set

$$\begin{aligned} w_j^+(\epsilon) &= W_\epsilon(\epsilon^{-1}tb_{j+1}^\epsilon) - \epsilon^{-1}q, \qquad j = 0, 1, \ldots, \rho(\epsilon) \\ w_j^-(\epsilon) &= W_\epsilon(\epsilon^{-1}tb_j) - \epsilon^{-1}q, \qquad j = 1, 2, \ldots, \rho(\epsilon) \end{aligned} \tag{5.32}$$

$$\begin{aligned} \gamma_j^+(\epsilon) &= \left[w_j^+(\epsilon)\right]^{3/4}, \qquad j = 0, 1, \ldots, \rho(\epsilon) \\ \gamma_j^-(\epsilon) &= \left[w_j^-(\epsilon)\right]^{3/4}, \qquad j = 1, 2, \ldots, \rho(\epsilon) \end{aligned} \tag{5.33}$$

It is not hard to check that for ϵ small enough the bound (5.19) is valid if we replace q by $\epsilon^{-1}q$, s by $s_j^+(\epsilon) = w_j^+(\epsilon) + [w_j^+(\epsilon)]^{7/8}$, γ by $\gamma_j^+(\epsilon)$, τ by

$\epsilon^{-1}t$, and a_1 by $v - b_{j+1}^{\epsilon}$ for all $j = 0, 1, \ldots, \rho(\epsilon)$. Similarly, for ϵ small enough, the bound (5.21) is valid if we replace q by $\epsilon^{-1}q$, s by $s_j^-(\epsilon) = w_j^-(\epsilon) - [w_j^-(\epsilon)]^{7/8}$, γ by $\gamma_j^-(\epsilon)$, τ by $\epsilon^{-1}t$, and a_2 by $v - b_j^{\epsilon}$ for all $j = 1, 2, \ldots, \rho(\epsilon)$. Hence the estimates (5.20) and (5.22) give

$$P\left(\left\{X \in \mathscr{M}_d' : \sum_{j=1}^{\rho(\epsilon)} \xi_{[\epsilon^{-1}q, \epsilon^{-1}q + s_j^-(\epsilon)] \times \Delta_j^\epsilon}(X) \leqslant \sum_{j=0}^{\rho(\epsilon)} n_{X,\Delta_j^\epsilon}^{+}(\epsilon^{-1}q, v, \epsilon^{-1}t)\right.\right.$$
$$\left.\left.\leqslant \sum_{j=0}^{\rho(\epsilon)} \xi_{[\epsilon^{-1}q, \epsilon^{-1}q + s_j^+(\epsilon)] \times \Delta_j^\epsilon}(X)\right\}\right)$$
$$\geqslant 1 - 0\left(\epsilon^{1/2} \ln^{3/2}(1/\epsilon)\right) \tag{5.34}$$

where the right-hand side can be taken independent of $q \in \mathbb{R}^1$ and $v \in J$. Hence it suffices to prove that both variables $\epsilon \sum_{j=1}^{\rho(\epsilon)} |X_{[\epsilon^{-1}q, \epsilon^{-1}q + s_j^-(\epsilon)] \times \Delta_j^\epsilon}|$ and $\epsilon \sum_{j=0}^{\rho(\epsilon)} |X_{[\epsilon^{-1}q, \epsilon^{-1}q + s_j^+(\epsilon)] \times \Delta_j^\epsilon}|$ tend to $m_{g_0}^{+}(q, v, t)$ in the sense of convergence in probability, uniformly in $(q, v) \in \mathbb{R}^1 \times J$.

Let us consider the "$-$" case. The other one is treated similarly. Using definitions (5.27), (5.27′), and (5.32) we see that the limiting behavior of

$$\epsilon \;{}_{P^\epsilon} \sum_{j=1}^{\rho(\epsilon)} \xi_{[\epsilon^{-1}q, \epsilon^{-1}q + s_j^-(\epsilon)] \times \Delta_j^\epsilon}$$

is the same as that of

$$\int_{v - \ln(1/\epsilon)} dv' \int_q^{\epsilon W_\epsilon(\epsilon^{-1}t(v - v'))} dq'\, k_P^{(1)}(\epsilon^{-1}q', v')$$

From definition (5.30) and assumption (ii) of Theorem 4.I, it follows that

$$\lim_{\epsilon \to 0} \epsilon W_\epsilon\left(\epsilon^{-1}t(v - v')\right) = B_{g_0,q}^{-1}(q + t(v - v'))$$

[see Eq. (3.8)]. Using again assumption (ii) of Theorem 4.I, we conclude that

$$\lim_{\epsilon \to 0} \epsilon \mathbb{E}_{P^\epsilon} \sum_{j=1}^{\rho(\epsilon)} \xi_{[\epsilon^{-1}q, \epsilon^{-1}q + s_j^-(\epsilon)] \times \Delta_j^\epsilon} = \int_{-\infty}^{v} dv' \int_q^{B_{g_0,q}^{-1}(q + t(v - v'))} dq'\, g_0(q', v')$$
$$= \int_{-\infty}^{v} dv' \int_q^{q + t(v - v')} dq'\, f_q(q', v')$$
$$= m_{g_0}^{+}(q, v, t) \tag{5.35}$$

Notice that the relation (5.35) is uniform in $(q, v) \in \mathbb{R}^1 \times J$.

Now the Chebyshev inequality together with property (E) gives

$$P^{\epsilon}\left(\left\{\left|\epsilon\sum_{j=1}^{\rho(\epsilon)}\xi_{[\epsilon^{-1}q,\epsilon^{-1}q+s_j^-(\epsilon)]\times\Delta_j^\epsilon}(X)-\epsilon\sum_{j=1}^{\rho(\epsilon)}\mathbb{E}_{P^\epsilon}\xi_{[\epsilon^{-1}q,\epsilon^{-1}q+s_j^-(\epsilon)]\times\Delta_j^\epsilon}\right|\right.\right.$$
$$\left.\left.\geqslant\epsilon\sum_{j=1}^{\rho(\epsilon)}\left[w_j^-(\epsilon)\right]^{3/4}\right\}\right)$$
$$\leqslant O\left(\epsilon^{1/2}\ln^{3/2}(1/\epsilon)\right) \tag{5.36}$$

where again the right-hand side is independent of $(q,v)\in\mathbb{R}^1\times J$. This concludes the proof of Eq. (5.29). Theorem 5.II is proved.

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